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Finite-Size Effects at Critical Points with Anisotropic Correlations: Phenomenological Scaling Theory and Monte Carlo Simulations

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Various thermal equilibrium and nonequilibrium phase transitions exist where the correlation lengths in different lattice directions diverge with different exponents $\nu_{||}$, $\nu_{\perp}$: uniaxial Lifshitz points, the Kawasaki spin exchange model driven by an electric field, etc. An extension of finite-size scaling concepts to such anisotropic situations is proposed, including a discussion of (generalized) rectangular geometries, with linear dimension $L_{||}$ in the special direction and linear dimensions $L_{\perp}$ in all other directions. The related shape effects for $L_{||} \neq L_{\perp}$ but isotropic critical points are also discussed. Particular attention is paid to the case where the generalized hyperscaling relation $\nu_{||} + (d-1)\nu_{\perp} = \gamma + 2\beta$ does not hold. As a test of these ideas, a Monte Carlo simulation study for shape effects at isotropic critical point in the two-dimensional Ising model is presented, considering subsystems of a 1024×1024 square lattice at criticality.

KEY WORDS: Finite-size scaling; anisotropic systems; Lifshitz points; driven Kawasaki model; nonequilibrium phase transitions; Monte Carlo simulations.

1. INTRODUCTION

Finite-size effects on phase transitions have been given a lot of attention in the last two decades[1-19] (refs. 7 and 16–18 contain recent reviews of some aspects of this work; see ref. 1 for early work). The most commonly studied cases are second-order phase transitions where, close enough to the critical temperature T_c, there is a single relevant length: this length scale is the

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correlation length ξ of the order parameter fluctuations, which diverges according to a power law involving the exponent ν,

$$\xi \propto |T-T_c|^{-\nu} \tag{1.1}$$

as one approaches T_c. The basic statement of finite-size scaling theory[2,4,5,7] is that the finite-size rounding and shifting of critical singularities are controlled by the ratio between ξ and the linear dimension L of the system. Thus, if we consider for the moment a (hyper-) cubic geometry $L \times L \times \cdots \times L$ in d dimensions, then the order parameter $\langle |\Psi| \rangle$ which in the thermodynamic limit $L \to \infty$ vanishes according to a power law involving the exponent β,

$$\langle |\Psi| \rangle \propto (T_c - T)^\beta, \qquad T \to T_c \tag{1.2}$$

has, in the finite geometry, a smooth temperature variation described by a scaling function $\tilde{\Psi}$,

$$\langle |\Psi| \rangle_L \simeq L^{-\beta/\nu} \tilde{\Psi}(L/\xi) \tag{1.3}$$

In a similar way the ordering susceptibility

$$\chi_\Psi \equiv \lim_{L\to\infty} \begin{cases} (L^d/k_B T)[\langle \Psi^2 \rangle_L - \langle |\Psi| \rangle_L^2], & T < T_c \\ (L^d/k_B T)\langle \Psi^2 \rangle_L, & T \geqslant T_c \end{cases} \tag{1.4}$$

which in the thermodynamic limit diverges according to a power law involving the critical exponent γ,

$$\chi_\Psi \propto |T-T_c|^{-\gamma}, \qquad T \to T_c \tag{1.5}$$

has its finite-size behavior given by

$$\chi_{\Psi,L} \simeq L^{\gamma/\nu} \tilde{\chi}(L/\xi) \tag{1.6}$$

Equations (1.3) and (1.6) are only valid if the hyperscaling relation among the critical exponents is satisfied, namely[20]

$$d\nu = \gamma + 2\beta \tag{1.7}$$

If Eq. (1.7) does not hold, the asymptotic critical behavior is not determinated by a single diverging length, and thus it is clear that the simple rule "L scales with ξ" expressed in Eqs. (1.3) and (1.6) may need modification. Several cases need to be distinguished:

(i) For systems above the marginal dimensionality d^*, the critical exponents take on their mean-field values and then $d\nu > \gamma + 2\beta$. In this case

Eqs. (1.3) and (1.6) are not valid[6,9–13] and the finite-size behavior near T_c is controlled by lengths different from ξ[10–12].

(ii) For certain anisotropic systems, the decay of the correlation functions in different directions may be governed by correlation lengths diverging with different exponents. Here we shall mainly consider uniaxial systems, with a correlation length $\xi_{\parallel}$ in the direction of this axis, and another correlation length $\xi_{\perp}$ in all directions perpendicular to this axis:

$$\xi_{\parallel} \propto |T-T_c|^{-\nu_{\parallel}} \tag{1.8}$$

$$\xi_{\perp} \propto |T-T_c|^{-\nu_{\perp}} \tag{1.9}$$

with $\nu_{\parallel}$ and $\nu_{\perp}$ different from each other. Such a case is encountered for uniaxial Lifshitz points,[21–23] for instance. In this case a modified hyperscaling relation holds,[21]

$$\nu_{\parallel} + (d-1)\nu_{\perp} = \gamma + 2\beta \tag{1.10}$$

if the system dimensionality d does not exceed the marginal dimensionality d^* ($d^* = 9/2$ for the uniaxial Lifshitz point[21]). A generalization to m-axial Lifshitz points (where the correlation length is $\xi_{\parallel}$ in m directions and $\xi_{\perp}$ in $d-m$ directions) is conceivable but outside of consideration here.

(iii) The situation described by Eqs. (1.8) and (1.9) is believed also to happen for the nonequilibrium phase transition which occurs in the Kawasaki spin exchange Ising model[24] driven by an "electric field." In this model one associates an electric charge with the particles (represented by down spins in the lattice gas interpretation of the Ising model).[25] Despite many computer simulation efforts,[25–33] the critical behavior of this model is far from understood: while some analyses suggest that Eqs. (1.8) and (1.9) with $\nu_{\parallel} \neq \nu_{\perp}$ are valid for both $d=2$ and $d=3$, Vallés and Marro[29] found $\nu \approx 0.55 \pm 0.2$ in two dimensions, assuming that there is only one exponent $\nu_{\parallel} = \nu_{\perp} = \nu$. And a similar result $\nu \approx 0.7$ is also given in ref. 32. On the other hand, a field-theoretic version of this model has been studied by Janssen and Schmittmann[34] and by Leung and Cardy,[35] who obtain in an expansion to all orders in $\varepsilon = 5-d$ the following critical exponents:

$$\begin{gathered}\nu_{\parallel} = 1 + \frac{\varepsilon}{6}, \qquad \nu_{\perp} = \frac{1}{2}, \qquad \beta = \frac{1}{2}, \qquad \gamma = 1 \\ \eta_{\parallel}^{\mathrm{RS}} = \frac{(\varepsilon-2)(3+\varepsilon)}{6+\varepsilon}, \qquad \eta_{\perp}^{\mathrm{RS}} = 1 + \frac{\varepsilon}{3}\end{gathered} \tag{1.11}$$

which are believed to be valid for $2 < d < 5$. It has not been possible so far to carry out a meaningful comparison of Eq. (1.11) with simulation results, since the latter are plagued by finite-size effects which are not fully understood. Since for $d > 2$ the exponents in Eq. (1.11) do not satisfy the generalized hyperscaling relation (1.10), rather complicated finite-size behavior may be expected.

In the present paper, we discuss the extension of finite-size scaling concepts to anisotropic critical phenomena described by Eqs. (1.8) and (1.9), and consider both the situation where the generalized hyperscaling relation (1.10) is valid and where it is not. In a case with $\nu_{\parallel} \neq \nu_{\perp}$, it is rather natural to consider the situation where the linear dimension $L_{\parallel}$ in the axial direction differs from the linear dimension $L_{\perp}$ in the other directions. Of course, such a geometry may be of interest also for the simpler situation of isotropic critical phenomena where a unique correlation length ξ exists. We shall therefore consider such shape effects also in the standard finite-size scaling case. Previous work has occasionally considered the rectangular geometry $L_{\parallel} \times L_{\perp}$ in $d = 2$[1,8,15,17] and the infinite strip geometry[7,9,12,14]; anisotropically diverging correlation lengths and their consequence for finite-size scaling have been discussed for directed percolation[36,37] and for directed self-avoiding walks[38] and related models.[39] Whenever possible we shall make contact with these earlier works.

In the next section we describe our phenomenological theory in detail, while Section 3 describes Monte Carlo simulations which have been performed to test shape effects in the standard two-dimensional Ising model in thermal equilibrium. A computer simulation study of the driven Kawasaki model will be given in a future publication. Section 4 gives some comments on previous work on this problem and summarizes our conclusions.

2. PHENOMENOLOGICAL FINITE-SIZE SCALING THEORY

2.1. Prelude

This section, which is necessarily somewhat speculative, contains our main ideas and makes many new predictions. For the sake of clarity, we have divided it into many subsections, considering the straightforward cases first, and the more complicated situations later. Thus, the next three subsections will be devoted to cases where hyperscaling or generalized hyperscaling still holds, while the simultaneous problem of hyperscaling violation and anisotropy is treated in the last subsections. For the sake of coherency of the presentation, we shall summarize the main points of finite-size scaling for isotropic critical phenomena without hyperscaling in Section 2.5. This approach, first derived in refs. 10–12, is then generalized to anisotropic equilibrium phase transitions, namely Lifshitz points for

$d > d^*$, in Section 2.7. The extension to the nonequilibrium driven Kawasaki models is attempted in Section 2.9.

2.2. Shape Effects on Finite-Size Scaling in the Two-Dimensional Ising Model

In this subsection we treat finite systems in rectangular geometry $L_{\|} \times L_{\perp}$. We start by considering the correlation function between the local order parameter $\Psi(x, y)$ at a point described by its Cartesian coordinates x, y and the origin, $\langle \Psi(0,0)\, \Psi(x, y)\rangle_T$. The finite-size scaling hypothesis states for the behavior of this correlation function near T_c

$$\langle \Psi(0,0)\, \Psi(x, y)\rangle_T \simeq (x^2+y^2)^{-\eta/2} f\left(\frac{x}{\xi}, \frac{y}{\xi}, \frac{\xi}{L_{\|}}, \frac{\xi}{L_{\perp}}\right) \tag{2.1}$$

where η is the critical exponent describing the decay of correlations right at T_c in the infinite system.[20] Since we are particularly interested in the behavior at T_c where ξ is infinite, we note that Eq. (2.1) then can be reduced to

$$\langle \Psi(0,0)\, \Psi(x, y)\rangle_{T_c} \simeq (x^2+y^2)^{-\eta/2} F\left(\frac{x}{L_{\perp}}, \frac{y}{L_{\perp}}, \frac{L_{\|}}{L_{\perp}}\right) \tag{2.2}$$

Equation (2.2) results from Eq. (2.1) by defining $f'(z_1 z_4, z_2 z_4, z_3^{-1} z_4, z_4) \equiv f(z_1, z_2, z_3, z_4)$ and setting

$$F\left(\frac{x}{L_{\perp}}, \frac{y}{L_{\perp}}, \frac{L_{\|}}{L_{\perp}}\right) = \lim_{z_4 \to \infty} f'\left(\frac{x}{L_{\perp}}, \frac{y}{L_{\perp}}, \frac{L_{\|}}{L_{\perp}}, z_4\right) \tag{2.3}$$

We first consider the case where the system with linear dimensions $L_{\|}, L_{\perp}$ is a subsystem of an infinite system, and wish to obtain the susceptibility by summing over all the correlations. Then the function F is independent of $L_{\|}, L_{\perp}$, of course; the "susceptibility"

$$k_B T\chi = \frac{1}{L_{\|} L_{\perp}} \sum_{x_1=1}^{L_{\|}} \sum_{y_1=1}^{L_{\perp}} \sum_{x_2=1}^{L_{\|}} \sum_{y_2=1}^{L_{\perp}} \langle \Psi(x_1, y_1)\, \Psi(x_2, y_2)\rangle_T \tag{2.4}$$

becomes, if we replace sums by integrals and choose the lattice spacing as unity,

$$k_B T\chi \approx \frac{2}{L_{\|} L_{\perp}} \int_0^{L_{\|}} dx_1 \int_0^{L_{\perp}} dy_1 \int_0^{L_{\|}-x_1} dx \int_0^{L_{\perp}-y_1} dy\, \langle \Psi(0,0)\, \Psi(x, y)\rangle_T \tag{2.5}$$

Using now $\langle \Psi(0,0)\, \Psi(x,y) \rangle_{T_c} \propto (x^2+y^2)^{-\eta/2}$, this becomes

$$k_B T_c \chi(T_c) \propto \frac{2}{L_{\|} L_{\perp}} \int_0^{L_{\|}} dx_1 \int_0^{L_{\perp}} dy_1 \int_0^{L_{\|}-x_1} dx \int_0^{L_{\perp}-y_1} dy\, (x^2+y^2)^{-\eta/2}$$

$$= 2(L_{\|} L_{\perp})^{\gamma/(2\nu)} \int_0^1 dx_1' \int_0^1 dy_1'$$

$$\times \int_0^{1-x_1'} dx' \int_0^{1-y_1'} dy' \left(\frac{L_{\|}}{L_{\perp}} x'^2 + \frac{L_{\perp}}{L_{\|}} y'^2 \right)^{-\eta/2} \tag{2.6}$$

where in the last step an obvious rescaling of all integration variables has been performed, and the scaling relation $\gamma/\nu = 2-\eta$ is used. It is seen that Eq. (2.6) has the form predicted by conformal invariance,[17]

$$k_B T_c \chi(T_c) \simeq (L_{\|} L_{\perp})^{\gamma/2\nu}\, \tilde{\chi}(L_{\|}/L_{\perp}) \tag{2.7}$$

and the scaling function $\tilde{\chi}$ has the symmetry $\tilde{\chi}(L_{\|}/L_{\perp}) = \tilde{\chi}(L_{\perp}/L_{\|})$, since the integration variables in Eq. (2.6) may be relabeled.

In the case where $L_{\|} \gg L_{\perp}$ it is straightforward to calculate $\tilde{\chi}(L_{\|}/L_{\perp})$ explicitly, since then the term $(L_{\perp}/L_{\|})\, y'^2$ is negligible in comparison with $(L_{\|}/L_{\perp}) x'^2$ in the integrand. We obtain

$$\chi(T_c) \propto (L_{\|} L_{\perp})^{1-\eta/2} (L_{\|}/L_{\perp})^{-\eta/2} = L_{\perp} L_{\|}^{1-\eta} = L_{\perp} L_{\|}^{\gamma/\nu - 1} \tag{2.8}$$

Next we discuss an estimate for the magnetization $\langle |\Psi| \rangle_{T_c}$. We argue that $\langle |\Psi| \rangle_{T_c}$ is of the same order as the root mean square magnetization $\langle \Psi^2 \rangle_{T_c}^{1/2}$,

$$\langle |\Psi| \rangle_{T_c} \propto (\langle \Psi^2 \rangle_{T_c})^{1/2} = \left[\frac{k_B T_c \chi(T_c)}{L_{\|} L_{\perp}} \right]^{1/2} \tag{2.9}$$

and hence

$$\langle |\Psi| \rangle_{T_c} \propto (L_{\|} L_{\perp})^{-\beta/2\nu}\, \tilde{\Psi}(L_{\|}/L_{\perp}) \tag{2.10}$$

where we have used the hyperscaling relation (1.7) with $d=2$, and the scaling function $\tilde{\Psi}$ behaves similar to $[\tilde{\chi}(L_{\|}/L_{\perp})]^{1/2}$. For $L_{\|}/L_{\perp}$ of order unity, both $\tilde{\chi}$ and $\tilde{\Psi}$ should also be of order unity. However, for $L_{\|} \gg L_{\perp}$ we should have

$$\langle |\Psi| \rangle_{T_c} \propto (L_{\|} L_{\perp})^{-\beta/2\nu} (L_{\|}/L_{\perp})^{-\eta/4} = L_{\|}^{-\beta/\nu} \tag{2.11}$$

A further quantity of interest is the reduced fourth-order cumulant,

$$g \equiv \frac{1}{2}\left(3 - \frac{\langle \Psi^4 \rangle}{\langle \Psi^2 \rangle^2} \right) \tag{2.12}$$

which at T_c is expected to scale as

$$g(T_c) \simeq \tilde{g}(L_{||}/L_\perp) \tag{2.13}$$

A discussion of four-spin correlation functions at T_c is required to determine the asymptotic behavior of the scaling function $\tilde{g}(L_{||}/L_\perp)$ for large (or small) $L_{||}/L_\perp$, but this problem is outside of our considerations here. Again the symmetry $\tilde{g}(L_{||}/L_\perp) = \tilde{g}(L_\perp/L_{||})$ must hold.

Next we consider a finite system with linear dimensions $L_{||}$, $L_\perp$ and periodic boundary conditions. Due to the full translational invariance of this problem, we have instead of Eq. (2.4),

$$k_B T\chi = \sum_{x=1}^{L_{||}} \sum_{y=1}^{L_\perp} \langle \Psi(1,1)\,\Psi(x,y)\rangle_T \tag{2.14}$$

In the continuum limit Eq. (2.5) gets replaced by

$$k_B T\chi \approx \int_0^{L_{||}} dx \int_0^{L_\perp} dy\, \langle \Psi(0,0)\,\Psi(x,y)\rangle_{T, L_{||}, L_\perp} \tag{2.15}$$

Using now Eq. (2.2), we find at T_c

$$k_B T\chi(T_c) \approx \int_0^{L_{||}} dx \int_0^{L_\perp} dy\, (x^2+y^2)^{-\eta/2}\, F\left(\frac{x}{L_\perp}, \frac{y}{L_\perp}, \frac{L_{||}}{L_\perp}\right) \tag{2.16}$$

where now a nontrivial function F appears. Rescaling the integration variables in Eq. (2.16) as in Eq. (2.6) yields

$$\begin{aligned} k_B T_c\chi(T_c) \approx{}& (L_{||}L_\perp)^{\gamma/2\nu} \int_0^1 dx' \\ &\times \int_0^1 dy' \left(\frac{L_{||}}{L_\perp}x'^2 + \frac{L_\perp}{L_{||}}y'^2\right)^{-\eta/2} F\left(\frac{L_{||}}{L_\perp}x', y', \frac{L_{||}}{L_\perp}\right) \end{aligned} \tag{2.17}$$

and hence

$$k_B T_c\chi(T_c) \simeq (L_{||}L_\perp)^{\gamma/2\nu}\, \tilde{\chi}(L_{||}/L_\perp) \tag{2.18}$$

Although Eq. (2.18) has the same form as Eq. (2.7), the asymptotic behavior for $L_{||} \gg L_\perp$ clearly is different. As is well known,[8] for $L_{||} \gg L_\perp$ the correlation function decays exponentially fast in the x direction:

$$\begin{aligned} F\left(\frac{L_{||}}{L_\perp}x', y', \frac{L_{||}}{L_\perp}\right) &\propto \left(x'^2\frac{L_{||}^2}{L_\perp^2} + y'^2\right)^{\eta/2} \exp\left(-x'\frac{L_{||}}{L_\perp}\right) \\ &= \left(\frac{x^2}{L_\perp^2} + \frac{y^2}{L_\perp^2}\right)^{\eta/2} \exp\left(-\frac{x}{L_\perp}\right) \end{aligned} \tag{2.19}$$

Actually in the decay constant there may be an amplitude factor, which for simplicity is suppressed here, as prefactors anyway are disregarded throughout. This equation expresses the fact that for $L_{\|} \gg L_{\perp}$ in the fully finite system the correlation length at T_c is limited by $L_{\perp}$, and this length $L_{\perp}$ governs the decay of correlations in the direction along the strip. Equation (2.17) then yields, putting $x'' = x' L_{\|}/L_{\perp}$,

$$\begin{aligned} k_B T_c \chi(T_c) &\approx (L_{\|} L_{\perp})^{\gamma/2\nu} \frac{L_{\perp}}{L_{\|}} \int_0^{\infty} dx'' \int_0^1 dy' \left(\frac{L_{\perp}}{L_{\|}}\right)^{-\eta/2} \exp(-x'') \\ &\propto (L_{\|} L_{\perp})^{\gamma/2\nu} \frac{L_{\perp}}{L_{\|}} \left(\frac{L_{\perp}}{L_{\|}}\right)^{-\eta/2} = L_{\perp}^{2-\eta} \end{aligned} \tag{2.20}$$

where we have replaced the upper limit $L_{\|}/L_{\perp}$ of the x'' integration by infinity. Thus, for very large $L_{\|}$, $\chi(T_c)$ in fact does not depend on $L_{\|}$. Using now once more Eq. (2.9) yields

$$\begin{aligned} \langle |\Psi| \rangle_{T_c} &\propto (L_{\|}^{-1} L_{\perp}^{1-\eta})^{1/2} \\ &= L_{\perp}^{-\beta/\nu} \left(\frac{L_{\perp}}{L_{\|}}\right)^{1/2}, \qquad L_{\|} \gg L_{\perp} \end{aligned} \tag{2.21}$$

Equation (2.13) also holds in this case, but now it is clear that the distribution of the order parameter Ψ in a system with $L_{\|} \gg L_{\perp}$ is a Gaussian even at T_c, consistent with the exponential vanishing of the correlation function, Eq. (2.19). Thus, $g(L_{\|} \gg L_{\perp}) \to 0$ as $L_{\|} \to \infty$. In fact one can show that g vanishes proportional to the inverse of the larger linear dimension of the system,[14]

$$g(T_c) \simeq (L_{\perp}/L_{\|})\, g^*, \qquad L_{\|} \gg L_{\perp} \tag{2.22}$$

where g^* is a (universal) constant which has been estimated from extrapolation of transfer matrix results and conformal invariance as[14] $g^* \approx 3.96$.

2.3. Shape Effects on Finite-Size Scaling in Two-Dimensional Anisotropic Models

As is well known, the standard lattice anisotropy where exchange constants $J_{\|}$ and $J_{\perp}$ are taken to be different in the two lattice directions induces an anisotropy only in the critical amplitudes of the correlation lengths $\xi_{\|}$ and $\xi_{\perp}$ in the two lattice directions, while the critical exponent ν remains the same in both directions and still retains its isotropic value.[40] With a suitable rescaling of the units of lengths in x and y directions,[40] Eq. (2.1) still holds and thus the analysis of Section 2.2 still applies. Such a situation will not be discussed further here.

Instead we are interested in a more essential kind of anisotropy where $\xi_{\|}$ and $\xi_{\perp}$ diverge with different exponents $\nu_{\|}, \nu_{\perp}$ [Eqs. (1.8), (1.9)] as T_c is approached. We now *assume* that a generalization of Eq. (2.1) holds, as follows:

$$\langle \Psi(0,0)\,\Psi(x,y)\rangle_{T_c} \simeq x^{-\eta_{\|}^{\mathrm{RS}}} f\left(\frac{x}{\xi_{\|}}, \frac{y}{\xi_{\perp}}, \frac{\xi_{\|}}{L_{\|}}, \frac{\xi_{\perp}}{L_{\perp}}\right) \tag{2.23}$$

Related scaling hypotheses were assumed for directed percolation,[36,37] while for directed self-avoiding walks this form of finite-size scaling does not hold[38] and is replaced by a weaker form of scaling, which will not be discussed at this point (but see Section 2.6 below). Right at T_c we eliminate $\xi_{\|}, \xi_{\perp}$ from the scaling function f by redefining it as

$$f(z_1, z_2, z_3, z_4) = f'(z_1 z_3, z_2 z_4, z_3^{-1} z_4^{\nu_{\|}/\nu_{\perp}}, z_4)$$

and defining its limiting function

$$\lim_{z_4 \to \infty} f' = F\left(\frac{x}{L_{\|}}, \frac{y}{L_{\perp}}, \frac{L_{\|}}{L_{\perp}^{\nu_{\|}/\nu_{\perp}}}\right) \tag{2.24}$$

Then Eq. (2.23) becomes

$$\langle \Psi(0,0)\,\Psi(x,y)\rangle_{T_c} \simeq x^{-\eta_{\|}^{\mathrm{RS}}} F\left(\frac{x}{L_{\|}}, \frac{y}{L_{\perp}}, \frac{L_{\|}}{L_{\perp}^{\nu_{\|}/\nu_{\perp}}}\right) \tag{2.25}$$

If we consider again a subsystem of an infinite system, F must in fact be independent of both $L_{\|}$ and $L_{\perp}$, which implies that $F(z_1, z_2, z_3) = \tilde{f}(z_1 z_2^{-\nu_{\|}/\nu_{\perp}} z_3)$, i.e.,

$$\langle \Psi(0,0)\,\Psi(x,y)\rangle_{T_c} \simeq x^{-\eta_{\|}^{\mathrm{RS}}} \tilde{f}(x/y^{\nu_{\|}/\nu_{\perp}}) \tag{2.26}$$

where $\tilde{f}(z \to \infty) \to \text{const}$ and

$$\tilde{f}(z \to 0) \propto z^{\eta_{\perp}^{\mathrm{RS}} \nu_{\perp}/\nu_{\|}} \tag{2.27}$$

in order that

$$\langle \Psi(0,0)\,\Psi(x,0)\rangle_{T_c} \propto x^{-\eta_{\|}^{\mathrm{RS}}}, \qquad \langle \Psi(0,0)\,\Psi(0,y)\rangle_{T_c} \propto y^{-\eta_{\perp}^{\mathrm{RS}}}$$

in the notation of ref. 35. Of course, the isotropic case ($\eta_{\|}^{\mathrm{RS}} = \eta_{\perp}^{\mathrm{RS}} = \eta$, $\nu_{\|} = \nu_{\perp} = \nu$) considered in Eq. (2.2) still is a special case of this description; one just has to take $\tilde{f}(x/y) = (1 + y^2/x^2)^{-\eta/2}$ in Eq. (2.26). In the anisotropic case, the condition that for $x \to 0$, $y \neq 0$ the factor $x^{-\eta_{\|}^{\mathrm{RS}}}$ cancels out yields a scaling law

$$\eta_{\|}^{\mathrm{RS}} \nu_{\|} = \eta_{\perp}^{\mathrm{RS}} \nu_{\perp} \tag{2.28}$$

A generalization of Eq. (2.28) for arbitrary dimensions d is

$$(d-2+\eta_{\|}^{\mathrm{RS}})\nu_{\|} = (d-2+\eta_{\perp}^{\mathrm{RS}})\nu_{\perp} \tag{2.29}$$

It is interesting to note that for the field-theoretic version of the driven Kawasaki model[34,35] Eq. (2.29) holds with the predicted exponents in Eq. (1.11) for $2 \leqslant d \leqslant 5$, while the generalized hyperscaling relation (1.10) does not hold except for $d=2$. Since at $d=2$ a variable which was "dangerously irrelevant" for $d>2$ (in a renormalization group sense[41]) becomes relevant, one should expect logarithmic corrections to the simple scaling behavior as it is assumed here. However, since we do not see any firm method to predict such logarithmic correction terms quantitatively, we disregard them throughout in our analysis. Thus, it is not clear to which model Eqs. (2.23)–(2.28) actually apply as they stand.

Accepting thus Eqs. (2.23)–(2.28) as a working hypothesis, we may still replace the sum in Eq. (2.4) by integrals [Eq. (2.5)], provided that $\eta_{\|}^{\mathrm{RS}} < 1$, $\eta_{\perp}^{\mathrm{RS}} < 1$, to obtain for the subsystem geometry

$$k_B T_c \chi(T_c) \approx \frac{2}{L_{\|} L_{\perp}} \int_0^{L_{\|}} dx_1 \int_0^{L_{\perp}} dy_1 \int_0^{L_{\|}-x_1} dx\, x^{-\eta_{\|}^{\mathrm{RS}}} \int_0^{L_{\perp}-y_1} dy\, \tilde{f}\left(\frac{x}{y^{\nu_{\|}/\nu_{\perp}}}\right) \tag{2.30}$$

which becomes after simple rescaling of integration variables

$$k_B T_c \chi(T_c) \approx 2L_{\perp} L_{\|}^{1-\eta_{\|}^{\mathrm{RS}}} \int_0^1 dx_1' \int_0^1 dy_1' \times \int_0^{1-x_1'} dx'\, x'^{-\eta_{\|}^{\mathrm{RS}}} \int_0^{1-y_1'} dy'\, \tilde{f}\left(\frac{x'}{y'^{\nu_{\|}/\nu_{\perp}}} \frac{L_{\|}}{L_{\perp}^{\nu_{\|}/\nu_{\perp}}}\right) \tag{2.31}$$

As a consequence, we predict

$$k_B T_c \chi(T_c) \approx L_{\perp} L_{\|}^{1-\eta_{\|}^{\mathrm{RS}}} \tilde{\chi}\left(\frac{L_{\|}}{L_{\perp}^{\nu_{\|}/\nu_{\perp}}}\right) = L_{\|}^{\nu_{\perp}/\nu_{\|}+1-\eta_{\|}^{\mathrm{RS}}} \tilde{\chi}'\left(\frac{L_{\|}}{L_{\perp}^{\nu_{\|}/\nu_{\perp}}}\right) \tag{2.32}$$

where $\tilde{\chi}$ and $\tilde{\chi}'$ are suitable scaling functions. Invoking the scaling law[35]

$$2-\eta_{\|} = \nu_{\perp}/\nu_{\|} + 1 - \eta_{\|}^{\mathrm{RS}} = \gamma/\nu_{\|} \tag{2.33}$$

one can express $\chi(T_c)$ also in terms of the exponent $\eta_{||}$ describing the variation of the wavevector-dependent susceptibility $\chi(k_{||}, T_c) \propto k_{||}^{-(2-\eta_{||})}$,

$$k_B T_c \chi(T_c) \simeq L_{||}^{2-\eta_{||}} \tilde{\chi}' \left(\frac{L_{||}}{L_{\perp}^{\nu_{||}/\nu_{\perp}}} \right) \tag{2.34}$$

Again we deduce the limiting behavior of $\chi(T_c)$ when $L_{||}$ is large or small in comparison with $L_{\perp}^{\nu_{||}/\nu_{\perp}}$. For $L_{||} \gg L_{\perp}^{\nu_{||}/\nu_{\perp}}$, it is basically $f(\infty)$, which matters since $x'/y'^{\nu_{||}/\nu_{\perp}}$ is mostly of order unity in the integration domain of the integral, Eq. (2.31). Thus we must have $\chi(T_c) \propto L_{\perp} L_{||}^{1-\eta_{||}^{\mathrm{RS}}}$ in this limit, which implies that $\tilde{\chi}'$ in Eqs. (2.32) and (2.34) behaves as

$$\tilde{\chi}'(z \to \infty) \propto z^{-\nu_{\perp}/\nu_{||}} \tag{2.35}$$

for the subsystem geometry. In the inverse limit, $L_{||} \ll L_{\perp}^{\nu_{||}/\nu_{\perp}}$, it is the behavior of Eq. (2.27) which matters, and hence

$$\begin{aligned} k_B T_c \chi(T_c) \approx{} & L_{\perp} L_{||}^{1-\eta_{||}^{\mathrm{RS}}} \int_0^1 dx_1' \int_0^1 dy_1' \int_0^{1-x_1'} dx'\, x'^{-\eta_{||}^{\mathrm{RS}}} x'^{-\eta_{\perp}^{\mathrm{RS}} \nu_{\perp}/\nu_{||}} \\ & \times \int_0^{1-y_1'} dy'\, y'^{-\eta_{\perp}^{\mathrm{RS}}} \left(\frac{L_{||}}{L_{\perp}^{\nu_{||}/\nu_{\perp}}} \right)^{\eta_{\perp}^{\mathrm{RS}} \nu_{\perp}/\nu_{||}} \end{aligned} \tag{2.36}$$

which yields

$$\chi(T_c) \propto L_{\perp} L_{||}^{1-\eta_{||}^{\mathrm{RS}}} \left(\frac{L_{||}}{L_{\perp}^{\nu_{||}/\nu_{\perp}}} \right)^{\eta_{||}^{\mathrm{RS}}} = L_{||} L_{\perp}^{1-\eta_{\perp}^{\mathrm{RS}}} \tag{2.37}$$

using once more Eq. (2.28). Due to Eqs. (2.28) and (2.33) we can rewrite these results in terms of only the exponents γ, $\nu_{\perp}$, and $\nu_{||}$:

$$\chi(T_c) \propto L_{\perp} L_{||}^{\gamma/\nu_{||} - \nu_{\perp}/\nu_{||}}, \qquad L_{||} \gg L_{\perp}^{\nu_{||}/\nu_{\perp}} \tag{2.38}$$

$$\chi(T_c) \propto L_{||}^{\gamma/\nu_{||}} \propto L_{\perp}^{\gamma/\nu_{\perp}}, \qquad L_{||} \approx L_{\perp}^{\nu_{||}/\nu_{\perp}} \tag{2.39}$$

$$\chi(T_c) \propto L_{||} L_{\perp}^{\gamma/\nu_{\perp} - \nu_{||}/\nu_{\perp}}, \qquad L_{||} \ll L_{\perp}^{\nu_{||}/\nu_{\perp}} \tag{2.40}$$

When $\nu_{||} = \nu_{\perp}$, Eqs. (2.38) and (2.40) are identical if one replaces $L_{||}$ by $L_{\perp}$. Then the symmetry property of the scaling function $\tilde{\chi}$ in the isotropic case, $\tilde{\chi}(L_{||}/L_{\perp}) = \tilde{\chi}(L_{\perp}/L_{||})$, is recovered, while in the anisotropic case there is no such symmetry.

Estimating once more order parameters from Eq. (2.9) yields a generalization of Eq. (2.11),

$$\langle |\Psi| \rangle_{T_c} \propto L_{||}^{-\beta/\nu_{||}}, \qquad L_{||} \gg L_{\perp}^{\nu_{||}/\nu_{\perp}} \tag{2.41}$$

$$\langle |\Psi| \rangle_{T_c} \propto L_{\perp}^{-\beta/\nu_{\perp}}, \qquad L_{||} \ll L_{\perp}^{\nu_{||}/\nu_{\perp}} \tag{2.42}$$

The obvious generalization of Eq. (2.13) is

$$g(T_c) \simeq \tilde{g}(L_{\|}/L_{\perp}^{\nu_{\|}/\nu_{\perp}}) \tag{2.43}$$

The case of the fully finite geometry ($L_{\|} \times L_{\perp}$, with periodic boundary conditions) is treated similarly, using Eq. (2.25) in Eq. (2.15), to obtain

$$k_B T_c \chi(T_c) \simeq \int_0^{L_{\|}} dx \int_0^{L_{\perp}} dy\, x^{-\eta_{\|}^{RS}} F\left(\frac{x}{L_{\|}}, \frac{y}{L_{\perp}}, \frac{L_{\|}}{L_{\perp}^{\nu_{\|}/\nu_{\perp}}}\right) \tag{2.44}$$

Equation (2.44) implies the same general scaling structure as found in Eq. (2.32),

$$\begin{aligned} k_B T_c \chi(T_c) &\simeq L_{\perp} L_{\|}^{1-\eta_{\|}^{RS}} \int_0^1 dx'\, x'^{-\eta_{\|}^{RS}} \int_0^1 dy'\, F\left(x', y', \frac{L_{\|}}{L_{\perp}^{\nu_{\|}/\nu_{\perp}}}\right) \\ &= L_{\perp} L_{\|}^{1-\eta_{\|}^{RS}} \tilde{\chi}\left(\frac{L_{\|}}{L_{\perp}^{\nu_{\|}/\nu_{\perp}}}\right) \end{aligned} \tag{2.45}$$

But now the behavior of the function F will again be governed by exponential decays with x' (if $L_{\|} \gg L_{\perp}^{\nu_{\|}/\nu_{\perp}}$) or with y' (if $L_{\|} \ll L_{\perp}^{\nu_{\|}/\nu_{\perp}}$) and only for $L_{\|} \approx L_{\perp}^{\nu_{\|}/\nu_{\perp}}$ do we have $\chi(T_c) \propto L_{\perp} L_{\|}^{1-\eta_{\|}^{RS}}$.
We thus put

$$\begin{aligned} (L_{\|} x')^{-\eta_{\|}^{RS}} F\left(x', y', \frac{L_{\|}}{L_{\perp}^{\nu_{\|}/\nu_{\perp}}} \to \infty\right) &\propto L_{\perp}^{-\eta_{\perp}^{RS}} \exp\left(-\frac{x}{L_{\perp}^{\nu_{\|}/\nu_{\perp}}}\right) \\ &= L_{\perp}^{-\eta_{\perp}^{RS}} \exp\left(-\frac{x' L_{\|}}{L_{\perp}^{\nu_{\|}/\nu_{\perp}}}\right) \end{aligned} \tag{2.46}$$

$$(L_{\|} x')^{-\eta_{\|}^{RS}} F\left(x', y', \frac{L_{\|}}{L_{\perp}^{\nu_{\|}/\nu_{\perp}}} \to 0\right) \propto L_{\|}^{-\eta_{\|}^{RS}} \exp\left(-\frac{y' L_{\perp}}{L_{\|}^{\nu_{\perp}/\nu_{\|}}}\right) \tag{2.47}$$

since a finite length $L_{\perp} \ll L_{\|}^{\nu_{\perp}/\nu_{\|}}$ introduces a decay constant $L_{\perp}^{\nu_{\|}/\nu_{\perp}}$ in the parallel direction, while in the inverse limit $L_{\|} \ll L_{\perp}^{\nu_{\|}/\nu_{\perp}}$ we expect a decay constant $L_{\|}^{\nu_{\perp}/\nu_{\|}}$ in the transverse direction, the correlation function being basically uniform in the other direction in each case. Equations (2.45)–(2.47), and (2.9) yield

$$\chi(T_c) \propto L_{\perp}^{\gamma/\nu_{\perp}}, \qquad \langle |\Psi| \rangle_{T_c} \propto L_{\perp}^{\gamma/2\nu_{\perp} - 1/2} L_{\|}^{-1/2}, \qquad L_{\|} \gg L_{\perp}^{\nu_{\|}/\nu_{\perp}} \tag{2.48}$$

$$\chi(T_c) \propto L_{\|}^{\gamma/\nu_{\|}}, \qquad \langle |\Psi| \rangle_{T_c} \propto L_{\|}^{\gamma/2\nu_{\|} - 1/2} L_{\perp}^{-1/2}, \qquad L_{\|} \ll L_{\perp}^{\nu_{\|}/\nu_{\perp}} \tag{2.49}$$

Note that again we have omitted possible constant prefactors in the arguments of the exponentials in both Eq. (2.46) and (2.47), since we are interested only in the exponents but not in the prefactors in the power laws, Eqs. (2.48) and (2.49).

2.4. Anisotropic Finite-Size Scaling in General Dimensions and the Relation to Generalized Hyperscaling

We now return to a d-dimensional uniaxially anisotropic system with a local order parameter $\Psi(x_1, \ldots, x_{d-1}, z)$, with the $d-1$ transverse coordinates x_j and one longitudinal coordinate z. For $T \geqslant T_c$ the correlation function is assumed to scale like

$$\langle \Psi(0, \ldots, 0)\, \Psi(x_1 = 0, \ldots, x_{d-1} = 0, z) \rangle_T \simeq z^{-(d-2+\eta_{\|}^{\mathrm{RS}})} \tilde{G}_{\|}\left(\frac{z}{\xi_{\|}}\right) \tag{2.50}$$

$$\langle \Psi(0, \ldots, 0)\, \Psi(x_1 = 0, \ldots, x_j, \ldots, x_{d-1} = 0, z = 0) \rangle_T \simeq x_j^{-(d-2+\eta_{\perp}^{\mathrm{RS}})} \tilde{G}_{\perp}\left(\frac{x_j}{\xi_{\perp}}\right), \qquad j = 1, \ldots, d-1 \tag{2.51}$$

consistent with the previous subsection. The fluctuation relation for the susceptibility [Eq. (1.4) or (2.4), respectively] now needs to be slightly generalized as

$$k_B T \chi = L_{\|} L_{\perp}^{d-1} (\langle \Psi^2 \rangle_T - \langle |\Psi| \rangle_T^2) \tag{2.52}$$

where the difference between $T < T_c$ [for which Eq. (2.52) holds as it stands] and $T > T_c$ [for which the term $\langle |\Psi| \rangle_T^2$ needs to be omitted, cf. Eq. (1.4)] will be disregarded, since it affects only prefactors (critical amplitudes) and not exponents, if hyperscaling holds.

It is tempting to generalize Eqs. (1.3) and (1.6) to the anisotropic situation as follows, motivated by Eq. (2.23):

$$\langle |\Psi| \rangle_T \simeq L_{\|}^{-\beta/\nu_{\|}} f_1(\xi_{\|}/L_{\|}, \xi_{\perp}/L_{\perp}) \tag{2.53}$$

$$k_B T \chi \simeq L_{\|}^{\gamma/\nu_{\|}} f_2(\xi_{\|}/L_{\|}, \xi_{\perp}/L_{\perp}) \tag{2.54}$$

with f_1, f_2 suitable scaling functions. It is clear that for $T = T_c$ Eqs. (2.53) and (2.54) are compatible with Eqs. (2.39), (2.41), or (2.49) of the previous subsection.

Now the generalized hyperscaling relation (1.10) is realized if we require, in the same spirit as in Eq. (2.9), that $\langle |\Psi| \rangle_T^2$ and $\langle \Psi^2 \rangle_T$ scale in the same way, i.e.,

$$\langle \Psi^2 \rangle_T \simeq L_{\|}^{-2\beta/\nu_{\|}} f_1'(\xi_{\|}/L_{\|}, \xi_{\perp}/L_{\perp}) \tag{2.55}$$

Now Eqs. (2.52), (2.53), and (2.55) yield

$$k_B T\chi \simeq L_\perp^{d-1} L_\parallel^{1-2\beta/\nu_\parallel}\{f_1'(\xi_\parallel/L_\parallel, \xi_\perp/L_\perp) - f_1^2(\xi_\parallel/L_\parallel, \xi_\perp/L_\perp)\}$$
$$= L_\perp^{d-1} L_\parallel^{1-2\beta/\nu_\parallel}(L_\perp/L_\parallel^{\nu_\perp/\nu_\parallel})^{-(d-1)} f_2(\xi_\parallel/L_\parallel, \xi_\perp/L_\perp) \quad (2.56)$$

In the last step we have split off the curly bracket in the first line of Eq. (2.56) a factor

$$[(L_\perp/\xi_\perp)/(L_\parallel/\xi_\parallel)^{\nu_\perp/\nu_\parallel}]^{1-d} \propto (L_\perp/L_\parallel^{\nu_\perp/\nu_\parallel})^{1-d}$$

constructed such that the temperature dependence cancels out and that it cancels the factor $L_\perp^{d-1}$. The remaining part of the scaling function is called f_2, since the result of this procedure

$$k_B T\chi \simeq L_\parallel^{1+(d-1)(\nu_\perp/\nu_\parallel)-2\beta/\nu_\parallel} f_2(\xi_\parallel/L_\parallel, \xi_\perp/L_\perp) \quad (2.57)$$

has precisely the form postulated in Eq. (2.54). Comparing the exponent $\gamma/\nu_\parallel$ of the power law prefactor $L_\parallel^{\gamma/\nu_\parallel}$ to the exponent of the power law prefactor in Eq. (2.57) yields the generalized hyperscaling law (1.10). This simple reasoning generalizes an analogous relation[5,16] between Eqs. (1.3), (1.6), and (1.7).

2.5. Modified Finite-Size Scaling in Isotropic Systems above their Marginal Dimension

In this section we consider Ising-like models at dimensionalities $d > d^*$, where the critical behavior hence is mean-field-like[20]: $\beta = 1/2$, $\gamma = 1$, $\nu = 1/2$, $d^* = 4$. Calculating the partition function $[\mathbf{r} = (x_1, ..., x_{d-1}, z) = (\mathbf{x}_\perp, z)]$,

$$Z = \int D\Psi(\mathbf{r}) \exp[-H_{\text{eff}}\{\Psi\}] \quad (2.58)$$

with a Ginzburg–Laudau-type Hamiltonian (u_0, t are constants),

$$H_{\text{eff}}\{\Psi\} = \int d\mathbf{r} \left\{\frac{1}{2}(\nabla\Psi)^2 + \frac{1}{2} t\Psi^2 + \frac{u_0}{4!}\Psi^4\right\} \quad (2.59)$$

the order parameter fluctuations can be treated as a perturbation in comparison with the mean-field behavior. Note that units of Ψ and of length have been defined such that there are no further constants multiplying the terms $\frac{1}{2}(\nabla\Psi)^2$ and $\frac{1}{2}t\Psi^2$.

In a finite system with one linear dimension $L_\parallel$ and $d-1$ linear dimensions $L_\perp$ and periodic boundary conditions in all directions, we can write

$$\Psi(\mathbf{x}_\perp, z) = \bar{\Psi} + \sum_{\mathbf{q}_\perp} \sum_{q_\parallel} \exp(i\mathbf{q}_\perp \cdot \mathbf{x}_\perp + iq_\parallel z)\, \delta\Psi_{\mathbf{q}_\perp, q_\parallel} \quad (2.60)$$

where the components of $\mathbf{q}_\perp$ are "quantized" in units of $2\pi/L_\perp$ and $q_{||}$ is quantized in units of $2\pi/L_{||}$. For the moment, we assume $L_\perp$ and $L_{||}$ to be of the same order. It can be shown that while the average order parameter mode $\bar{\Psi}$ cannot be treated perturbatively, all the fluctuations $\delta\Psi_{\mathbf{q}_\perp, q_{||}}$ are separated by a finite gap from this mode $\bar{\Psi}$, and can be integrated out.(12) Basically the finite-size effects are understood(10–12) by writing the partition function as

$$Z = \int d\bar{\Psi} \exp[-H(\bar{\Psi})] \tag{2.61}$$

$$\bar{H}(\bar{\Psi}) = L_{||} L_\perp^{d-1} \left[\frac{1}{2} t \bar{\Psi}^2 + \frac{u_0}{4!} \bar{\Psi}^4 \right] \tag{2.62}$$

which means we work with an order parameter probability distributions function $P_{L_{||}, L_\perp}(\Psi)$ in a finite system as follows:

$$P_{L_{||}, L_\perp}(\Psi) \propto \exp\left[-\left(\frac{1}{2} t \Psi^2 + \frac{u_0}{4!} \Psi^2 \right) L_{||} L_\perp^{d-1} \right] \tag{2.63}$$

From Eq. (2.63), the behavior at T_c (i.e., $t=0$) is straightforwardly obtained as

$$\langle |\Psi| \rangle_{T_c} = \frac{\int_0^\infty \Psi \exp[-(u_0/4!)\Psi^4 L_{||} L_\perp^{d-1}]\, d\Psi}{\int_0^\infty \exp[-(u_0/4!)\Psi^4 L_{||} L_\perp^{d-1}]\, d\Psi} \propto (L_{||} L_\perp^{d-1})^{-1/4} \tag{2.64}$$

$$\chi(T_c) \propto (L_{||} L_\perp^{d-1})^{1/2} \tag{2.65}$$

and $g(T_c)$ [Eq. (2.12)] is a constant which has been calculated.(12) It is seen that for $L_{||} \approx L_\perp$, Eq. (2.64) agrees with Eq. (1.3) ($\langle |\Psi| \rangle_{T_c} \propto L_{||}^{-\beta/\nu} \propto L_{||}^{-1}$ with mean-field exponents) only for $d=4$, and similarly Eq. (2.65) agrees with Eq. (1.6) [$\chi(T_c) \propto L_{||}^{\gamma/\nu} \propto L_{||}^2$ with mean-field exponents] for $d=4$, but not for $d>4$.[4]

It turns out, however, that finite-size effects for $d>4$ can be understood in terms of a simple modified form of finite-size scaling, where $L_{||}$ and $L_\perp$ are not scaled with the correlation length ξ but with a "thermodynamic length" l.(10,11) This is recognized most simply from Eq. (2.63) by noting that away from T_c Eq. (2.63) can be written, considering the limit $L_{||} \to \infty$, $L_\perp \to \infty$, in the form

$$P_{L_{||}, L_\perp}(\Psi) \propto \exp\left[-\frac{\Psi^2}{2k_B T \chi_>} L_{||} L_\perp^{d-1} \right], \qquad T > T_c \tag{2.66}$$

[4] At $d=4$ logarithmic correction factors to these power laws occur(13,19) which are outside of consideration here.

where $k_B T\chi_> = t^{-1}$, and

$$P_{L_\parallel, L_\perp}(\Psi) \propto \exp\left[-\frac{(\Psi - \Psi_{sp})^2}{2k_B T\chi_<} L_\parallel L_\perp^{d-1}\right] + \exp\left[-\frac{(\Psi + \Psi_p)^2}{2k_B T\chi_<} L_\parallel L_\perp^{d-1}\right], \qquad T < T_c \tag{2.67}$$

with $\Psi_{sp} = (-6t/u_0)^{1/2}$ and $k_B T\chi_< = (t + u_0 \Psi_{sp}^2/2)^{-1} = (-2t)^{-1}$. Both Eqs. (2.66) and (2.67) can be considered as limiting forms of a general scaling expression

$$P_{L_\parallel, L_\perp}(\Psi) \simeq |t|^{-\beta} \tilde{P}\left(\Psi |t|^{-\beta}, \frac{L_\parallel}{l}, \frac{L_\perp}{l}\right) \tag{2.68}$$

where the "thermodynamic length" l is defined as[10,11] [$k_B T\chi_> = \Gamma_+ t^{-\gamma}$, $k_B T\chi_< = \Gamma_-(-t)^{-\gamma}$; in the present case, $\Psi_{sp} = \hat{B}(-t)^\beta$, with critical amplitudes Γ_+, Γ_-, $\hat{B}$]

$$l \equiv (k_B T_c \Gamma_+)^{1/d} |t|^{-(2\beta+\gamma)/d} \propto |t|^{-\tilde{\nu}}, \qquad \tilde{\nu} = \frac{2\beta + \gamma}{d} \tag{2.69}$$

Then Eqs. (2.66) and (2.67) simply become, in terms of the scaled order parameter $\tilde{\Psi} \equiv \Psi |t|^{-\beta}$,

$$P_{L_\parallel, L_\perp}(\Psi) \propto \exp[-\tfrac{1}{2}\tilde{\Psi}^2 (L_\parallel/l)(L_\perp/l)^{d-1}], \qquad t > 0 \tag{2.70}$$

$$P_{L_\parallel, L_\perp}(\Psi) \propto \exp[-\tfrac{1}{2}(\tilde{\Psi} - \hat{B})^2 (\Gamma_+/\Gamma_-)(L_\parallel/l)(L_\perp/l)^{d-1}] + \exp[-\tfrac{1}{2}(\tilde{\Psi} + \hat{B})^2 (\Gamma_+/\Gamma_-)(L_\parallel/l)(L_\perp/l)^{d-1}], \qquad t < 0 \tag{2.71}$$

Comparing Eqs. (2.63) and (2.68), it is obvious that Eq. (2.68) holds in the mean-field region for $d > d^*$, with $\gamma = 1$, $\beta = 1/2$, and $\tilde{\nu} = 2/d$ then. But Eq. (2.68) also holds in the nontrivial critical region for $d < 4$, where hyperscaling holds and $\tilde{\nu} = \nu$. In fact, the fluctuation relation (2.66) is always true in the disordered phase for the considered limit, and also Eq. (2.67) holds provided Ψ is close to either Ψ_{sp} or to $-\Psi_{sp}$, and $L_\parallel$ and $L_\perp$ are of the same order. A very different behavior will be found when $L_\parallel \gg L_\perp$, however. This is the case to be studied in the next subsection.

2.6. Shape Effects on Finite-Size Scaling in Isotropic Systems above Their Marginal Dimension

When we consider a geometry with $L_\parallel \gg L_\perp$, even in the mean-field limit it is no longer legitimate to replace the functional integral (2.58) by the simple integral (2.61), where all fluctuations $\delta\Psi_{\mathbf{q}_\perp, q_\parallel}$ are neglected in comparison with the homogeneous order parameter $\bar{\Psi}$. While it is still appropriate to neglect the transverse fluctuations, inhomogeneities in the longitudinal direction must be allowed for. Thus, it is convenient to introduce a Fourier transform in the transverse directions only,

$$\Psi(\mathbf{x}_\perp, z) = \Psi(z) + \sum_{\mathbf{q}_\perp \neq 0} \exp(i\mathbf{q}_\perp \cdot \mathbf{x}_\perp)\, \delta\Psi_{\mathbf{q}_\perp}(z) \tag{2.72}$$

Still the components of $\mathbf{q}_\perp$ are quantized in unit of $2\pi/L_\perp$. The "dangerous" mode which cannot be treated perturbatively corresponds to $\mathbf{q}_\perp = 0$, and $\Psi(z)$ in Eq. (2.72) simply means $\delta\Psi_{\mathbf{q}_\perp=0}(z) + \bar{\Psi}$. The modes with $\mathbf{q}_\perp \neq 0$ are separated by a finite gap from this mode; they can be treated perturbatively and are integrated out. Zinn-Justin and Brézin[12] show that this type of perturbation theory converges for dimensionalities d exceeding the marginal dimension d^*; the perturbative terms yield *corrections* to finite-size scaling only.

We shall not repeat any details of this calculation[12] here, but we explore the fact that for $L_\parallel \gg L_\perp$ the partition function Z in Eq. (2.58) can be essentially written as a functional integral with a Hamiltonian for a one-dimensional problem,

$$Z = \int D\Psi(z) \exp[-H_{\text{eff}}\{\Psi\}] \tag{2.73}$$

$$H_{\text{eff}}\{\Psi\} = L_\perp^{d-1} \int_0^{L_\parallel} dz \left[\frac{1}{2}\left(\frac{d\Psi}{dz}\right)^2 + \frac{1}{2} t\Psi^2 + \frac{u_0}{4!}\Psi^4\right] \tag{2.74}$$

Since the system described by Eqs. (2.73), (2.74) is basically quasi-one-dimensional, the crucial quantity of interest is the correlation function $\langle \Psi(0)\, \Psi(z)\rangle_{L_\perp, T}$ and associated correlation length $\xi(L_\perp, T)$,

$$\langle \Psi(0)\, \Psi(z)\rangle_{L_\perp, T} \approx A(L_\perp, T) \exp\left[-\frac{z}{\xi(L_\perp, T)}\right], \qquad L_\perp \ll z \ll L_\parallel \tag{2.75}$$

Since Eqs. (2.73) and (2.74) describe essentially a field theory of the one-dimensional Ising model, it is clear that the correlation function (2.75) should have just a constant amplitude factor $A(L_\perp, T)$ in front of the exponential; no power law prefactor such as contained in Eq. (2.50) should appear. If we know $A(L_\perp, T)$ and $\xi(L_\perp, T)$ in Eq. (2.75), we can infer the finite-size behavior of the susceptibility as

$$\begin{aligned} k_B T\chi &= L_\perp^{d-1} \int_0^{L_\parallel} \langle \Psi(0)\, \Psi(z)\rangle_{L_\perp, T}\, dz \\ &\approx A(L_\perp, T)\, L_\perp^{d-1} \xi(L_\perp, T) \left\{1 - \exp\left[-\frac{L_\parallel}{\xi(L_\perp, T)}\right]\right\} \end{aligned} \tag{2.76}$$

Assuming once more that the order parameter $\langle|\Psi|\rangle$ is of the same order as the root mean square order parameter $\langle\Psi^2\rangle^{1/2}=[k_B T\chi/(L_{\|}L_{\perp}^{d-1})]^{1/2}$, we obtain

$$\langle|\Psi|\rangle \propto [A(L_{\perp},T)]^{1/2}\left[\frac{\xi(L_{\perp},T)}{L_{\|}}\right]^{1/2}\left\{1-\exp\left[-\frac{L_{\|}}{\xi(L_{\perp},T)}\right]\right\}^{1/2} \tag{2.77}$$

We now estimate both $\xi(L_{\perp},T)$ and $A(L_{\perp},T)$ from simple dimensional analysis arguments: the scaling behavior of the theory can thus be found without the need to actually evaluate the functional integral (2.73) precisely. Our reasoning is as follows.

For $t\geqslant 0$ (we consider here $T\geqslant T_c$ only) the minimum of $H_{\text{eff}}(\Psi)$ is given by $H_{\text{eff}}(\Psi_{\min})=0$, with $\Psi_{\min}(z)=0$. Now only such paths $\Psi(z)$ will contribute to the path integral (2.73) significantly for which $H_{\text{eff}}(\Psi)$ is of order unity and thus are sufficiently "close" to $\Psi_{\min}(z)$. Paths $\Psi(z)$ for which $H_{\text{eff}}(\Psi)$ is much larger than unity have negligible statistical weight.

We consider "typical" paths of the trial function form $\Psi(z)=\Psi_0 \exp(-z/\xi)$ consistent with a correlation function of the form (2.75). Here Ψ_0 and ξ have to be chosen such that $H_{\text{eff}}(\Psi)$ is of order unity for such a typical path. Apart from constants of order unity, these terms in the bracket of Eq. (2.74) contribute to $H_{\text{eff}}(\Psi)$ the terms

$$L_{\perp}^{d-1}\Psi_0^2/\xi, \qquad L_{\perp}^{d-1}t\Psi_0^2\xi, \qquad L_{\perp}^{d-1}u_0\Psi_0^4\xi \tag{2.78}$$

which for $t\geqslant 0$ are all nonnegative. For T sufficiently above T_c the first two terms actually dominate, and for the typical paths *both* of them should give contributions of the same order, namely the order unity. Putting all proportionality constants equal to unity for simplicity, we thus obtain

$$L_{\perp}^{d-1}\Psi_0^2/\xi=L_{\perp}^{d-1}t\Psi_0^2\xi=1 \tag{2.79}$$

Equation (2.79) yields the classical mean-field result $\xi\propto t^{-1/2}$ and $\Psi_0^2\propto t^{-1/2}/L_{\perp}^{d-1}$ and since $A(L_{\perp},T)$ is proportional to Ψ_0^2, we get from Eq. (2.76)

$$\chi\propto t^{-1}[1-\exp(-L_{\|}/t^{-1/2})]\cong t^{-1}$$

Thus, for T sufficiently above T_c the dimensional analysis method outlined above recovers the standard mean-field results, as it should.

More interesting is the behavior right at T_c, where the second term in Eq. (2.78) vanishes. Now we have to equate the first and the third terms in Eq. (2.78) to each other and to unity to characterize a typical path:

$$L_{\perp}^{d-1}\Psi_0^2/\xi_0=L_{\perp}^{d-1}u_0\Psi_0^4\xi_0=1 \tag{2.80}$$

from which the correlation length at $t=0$ (called ξ_0) follows as

$$\xi_0 = u_0^{-1/3} L_\perp^{(d-1)/3} \tag{2.81}$$

and $\Psi_0^2 = (u_0 \xi_0^2)^{-1}$. Equation (2.81) is a special case of Eq. (4.11) of Zinn-Justin and Brézin,[12] who quote a general scaling form describing the crossover from $\xi(L_\perp, T) \propto t^{-1/2}$ to ξ_0,

$$\xi(L_\perp, T) \simeq t^{-1/2} \tilde{\xi}\left(\frac{L_\perp}{t^{-1/2}} L_\perp^{(d-4)/3}\right), \qquad \tilde{\xi}(\zeta \to \infty) \propto \zeta \tag{2.82}$$

The fact that such a scaling must hold is simply anticipated from Eq. (2.74), which can be rescaled (for $L_\| \to \infty$) by defining z', Ψ' via

$$z' = u_0^{-1/6} \xi_0 z, \qquad \Psi' = \Psi u_0^{1/6} / \xi_0 \tag{2.83}$$

which yields

$$H(\Psi') = u_0^{-2} L_\perp^{2(d-1)} \int_0^\infty dz' \left\{ \frac{1}{2} \left(\frac{d\Psi'}{dz'} \right)^2 + \frac{1}{2} t u_0^{-2/3} L_\perp^{2(d-1)/3} \Psi'^2 + \frac{u_0}{4!} \Psi|^4 \right\} \tag{2.84}$$

implying

$$\xi(L_\perp, T) \simeq u_0^{-1/3} L_\perp^{(d-1)/3} \tilde{\tilde{\xi}}(t u_0^{-2/3} L_\perp^{2(d-1)/3}) \tag{2.85}$$

which is equivalent to Eq. (2.82). Zinn-Justin and Brézin[12] obtained this result from giving Eq. (2.74) a quantum mechanical interpretation, the term $\frac{1}{2}(d\Psi/dz)^2$ being interpreted as resulting from the kinetic energy term in the Feynman path integral representation. We here give a detailed rederivation of their results in slightly different terms, since we wish to generalize our method to cases where there is no obvious kinetic energy interpretation of the term replacing the gradient energy term.

Since the amplitude $A(L_\perp, T_c)$ in Eq. (2.75) is basically proportional to $\Psi_0^2 = u_0^{-1/3} L_\perp^{-2(d-1)/3}$, we find from Eqs. (2.76), (2.77) for $L_\| \to \infty$

$$k_B T_c \chi(T_c) \propto L_\perp^{2(d-1)/3} = L_\perp^{d/2} L_\perp^{(d-4)/6}, \qquad L_\| \to \infty \tag{2.86}$$

$$\langle |\Psi| \rangle_{T_c} \propto L_\perp^{-(d-1)/6} L_\|^{-1/2} = (L_\| L_\perp)^{-1/2} L_\perp^{-(d-4)/6}, \qquad L_\| \to \infty \tag{2.87}$$

For $L_\|$ finite, Eqs. (2.75)–(2.77) imply that $L_\|$ scales with $\xi_0 = \xi(L_\perp, T_c) \propto L_\perp^{(d-1)/3}$,

$$k_B T_c \chi(T_c) \simeq L_\perp^{d/2} L_\perp^{(d-4)/6} \tilde{\chi}\left(\frac{L_\perp}{L_\|} L_\perp^{(d-4)/3}\right) \tag{2.88}$$

$$\langle |\Psi| \rangle_{T_c} \simeq (L_\| L_\perp)^{-1/2} L_\perp^{-(d-4)/6} \tilde{\Psi}\left(\frac{L_\perp}{L_\|} L_\perp^{(d-4)/3}\right) \tag{2.89}$$

where the explicit form of the scaling functions $\tilde{\chi}$, $\tilde{\Psi}$ follows from the curly bracket in Eqs. (2.76), (2.77). We have $\tilde{\chi}(\zeta \to 0) = \text{const}$, $\tilde{\Psi}(\zeta \to 0) = \text{const}$, reproducing Eqs. (2.86), (2.87), while in the inverse limit we recover

$$k_B T_c \chi(T_c) \approx A(L_\perp, T_c) L_\perp^{d-1} L_{||} \propto L_\perp^{(d-1)/3} L_{||}$$

This happens when $L_{||}$ is of the order of $L_\perp^{(d-1)/3}$, and then $k_B T_c \chi(T_c) \propto L_\perp^{2(d-1)/3}$. On the other hand, for $L_{||} \leqslant L_\perp^{(d-1)/3}$ the correlation function (2.75) implies that Ψ is essentially homogeneous over the finite system, and hence the treatment of the previous section applies: using Eq. (2.65), $\chi(T_c) \propto (L_{||} L_\perp)^{1/2}$ for $L_{||} = L_\perp^{(d-1)/3}$, we in fact obtain $\chi(T_c) \propto L_\perp^{2(d-1)/3}$, which shows that for $L_{||}$ of the order of $L_\perp^{(d-1)/3}$ a smooth crossover between Eq. (2.65) and Eq. (2.88) occurs. Of course, Eqs. (2.75)–(2.77) cannot be used for $L_{||} \leqslant L_\perp^{(d-1)/3}$, since then the discreteness of $q_{||}$ in Eq. (2.60) needs to be taken into account, and one must no longer treat z as a continuous variable in Eq. (2.74). Thus, Eqs. (2.76) and (2.77) describe the scaling functions $\tilde{\chi}(\zeta)$, $\tilde{\Psi}(\zeta)$ in Eqs. (2.88), (2.89) only for $\zeta \leqslant 1$. The behavior of the scaling function for large ζ simply is found from Eqs. (2.64) and (2.65), namely

$$\tilde{\chi}(\zeta \to \infty) \propto \zeta^{-1/2}, \qquad \tilde{\Psi}(\zeta \to \infty) \propto \zeta^{-1/4} \tag{2.90}$$

Let us also discuss the generalization of Eqs. (2.88) and (2.89) to non-critical temperatures. Since at T_c, $L_{||}$ scales with ξ_0, Eq. (2.81), we now simply have to scale $L_{||}$ with $\xi(L_\perp, T)$ as obtained in Eq. (2.85) or (2.82), respectively. At the same time, however, we must include a second argument $L_\perp / t^{-2/d}$ in the scaling function, representing the scaling with the "thermodynamic length" which takes over in the case $L_\perp = L_{||}$. Thus, we conclude

$$\begin{aligned} &k_B T \chi(T, L_\perp, L_{||}) \\ &\quad \simeq L_\perp^{d/2} L_\perp^{(d-4)/6} \tilde{\tilde{\chi}} \left(\frac{L_\perp}{t^{-2/d}}, \frac{t^{-1/2}}{L_{||}} \tilde{\xi} \left(\frac{L_\perp}{t^{-1/2}} L_\perp^{(d-4)/3} \right) \right) \end{aligned} \tag{2.91}$$

$$\begin{aligned} &\langle |\Psi| \rangle_{T, L_\perp, L_{||}} \\ &\quad \simeq (L_{||} L_\perp)^{-1/2} L_\perp^{-(d-4)/6} \tilde{\tilde{\Psi}} \left(\frac{L_\perp}{t^{-2/d}}, \frac{t^{-1/2}}{L_{||}} \tilde{\xi} \left(\frac{L_\perp}{t^{-1/2}} L_\perp^{(d-4)/3} \right) \right) \end{aligned} \tag{2.92}$$

Equations (2.91) and (2.92) manifestly exhibit that there is no simple finite-size scaling for $d > d^* = 4$. Three different lengths come into play, the bulk correlation length ($\xi_b \propto t^{-\nu} = t^{-1/2}$), the thermodynamic length ($l \propto t^{-\bar{\nu}} = t^{-2/d}$), and the "longitudinal length" $\xi(L_\perp, T)$, which crosses over from ξ_b to $\xi_0 \propto L_\perp^{(d-1)/3}$ as $t \to 0$.

We emphasize at this point that all our considerations in this section refer to systems with periodic boundary conditions. The case of free boundary condition for an Ising-like system for $d > d^* = 4$ was considered by Rudnick *et al.*[42] They found that one has to replace the reduced temperature $t = (T - T_c)/T_c$ by a shifted reduced temperature defined by

$$\tilde{t} = t + \Gamma_+ \pi[(d-1)/L_\perp^2 + 1/L_\parallel^2] \tag{2.93}$$

where Γ_+ is the constant in the law $k_B T\chi(T \to T_c^+, L_\perp \to \infty, L_\parallel \to \infty) = L_+ t^{-1}$. We suggest that Eqs. (2.91) and (2.92) hold for free boundary conditions if t is replaced by $\tilde{t}$. It is clear that a different behavior occurs for subsystems of size $L_\perp^{d-1} L_\parallel$ of an infinite system: there we still can invoke an approach analogous to Eq. (2.4), namely

$$\begin{aligned} k_B T\chi = \frac{1}{L_\parallel L_\perp^{d-1}} \sum_{x_1=1}^{L_\perp} \cdots \sum_{x_{d-1}=1}^{L_\perp} \sum_{z=1}^{L_\parallel} \\ \times \sum_{x_1'=1}^{L_\perp} \cdots \sum_{x_{d-1}'=1}^{L_\perp} \sum_{z'=1}^{L_\parallel} \langle \Psi(x_1, ..., x_{d-1}, z)\, \Psi(x_1', ..., x_{d-1}', z') \rangle \end{aligned} \tag{2.94}$$

and use the mean-field correlation function, which behaves as follows:

$$\langle \Psi(0)\, \Psi(\mathbf{r}) \rangle \simeq r^{-(d-2+\eta)} G(r/\xi_b) \tag{2.95}$$

with $\eta = 0$ and the scaling function becoming $G(z) \propto \exp(-z)$ for large z. From Eqs. (2.94) and (2.95) we conclude

$$\begin{aligned} k_B T_c \chi(T_c) \propto \frac{1}{L_\parallel L_\perp^{d-1}} \int_0^{L_\perp} dx_1' \cdots \int_0^{L_\perp} dx_{d-1}' \int_0^{L_\parallel} dz' \int_0^{L_\perp - x_1'} dx_1 \cdots \\ \times \int_0^{L_\perp - x_{d-1}'} dx_{d-1} \int_0^{L_\parallel - z'} dz\, (x_1^2 + \cdots + x_{d-1}^2 + z^2)^{-(d-2)/2} \end{aligned} \tag{2.96}$$

and rescaling variables, this becomes

$$\begin{aligned} k_B T_c \chi(T_c) \propto L_\parallel L_\perp \int_0^1 dx_1' \cdots \int_0^1 dx_{d-1}' \int_0^1 dz' \int_0^{1-x_1'} dx_1 \cdots \\ \times \int_0^{1-x_{d-1}'} dx_{d-1} \int_0^{1-x'} dz \\ \times \left[x_1^2 + \cdots + x_{d-1}^2 + \left(\frac{L_\parallel^2}{L_\perp^2}\right) z^2 \right]^{-(d-2)/2} \end{aligned} \tag{2.97}$$

It is seen that for $L_{\|} \ll L_{\perp}$ as well as for $L_{\|} \approx L_{\perp}$ the result simply is of the order of

$$k_B T_c \chi(T_c) \propto L_{\|} L_{\perp} \tag{2.98}$$

in contrast to Eq. (2.65) (which holds under the same conditions on $L_{\|}$, $L_{\perp}$ for periodic boundary conditions and for free boundary conditions,[42] respectively). The result that different boundary conditions yield different power laws in the finite-size dependence at T_c is another signature of the breakdown of finite-size scaling in the standard form.

At this point, we recall that for directed random walks which show rather special finite-size scaling properties[38] the upper critical dimension is $d^* = 1$.

While most of the new results in this section [such as Eqs. (2.88)–(2.92)] are fairly straightforward extensions of known results,[12] the full power of our approach will become evident in the next subsections where we consider other models.

2.7. Isotropic Lifshitz Points

For an isotropic Lifshitz point,[21–23] the coefficient of the gradient term vanishes and thus $H_{\text{eff}}(\Psi)$ is no longer given by Eq. (2.59); instead a term $(\nabla^2 \Psi)^2$ needs to be considered,

$$H_{\text{eff}}\{\Psi\} = \int d\mathbf{r} \left[\frac{1}{2} (\nabla^2 \Psi)^2 + \frac{1}{2} t \Psi^2 + \frac{u_0}{4!} \Psi^4 \right] \tag{2.99}$$

Again units of Ψ and of lengths have been defined such that there are no further constants multiplying the terms $\frac{1}{2}(\nabla^2\Psi)^2$ and $\frac{1}{2}t\Psi^2$. While now the marginal dimension d^* [where Eq. (1.7) is satisfied with mean-field exponents] is $d^*_{\text{IL}} = 8$, since $\nu = 1/4$ in the mean field theory, the treatment of Section 2.5 still holds for systems with $d > d^*_{\text{IL}}$ with $L_{\|}$ and $L_{\perp}$ being of the same order. However, an interesting distinction occurs for systems having very anisotropic shapes, $L_{\|} \gg L_{\perp}$. Now the same reasoning as expressed in Eqs. (2.72) and (2.73) applies, but Eq. (2.74) gets replaced by

$$H_{\text{eff}}(\Psi) = L_{\perp}^{d-1} \int_0^{L_{\|}} dz \left[\frac{1}{2} \left(\frac{d^2 \Psi}{dz^2} \right)^2 + \frac{1}{2} t \Psi^2 + \frac{u_0}{4!} \Psi^4 \right] \tag{2.100}$$

While it is not obvious how to interpret this in terms of Schrödinger quantum mechanics and thus we cannot apply the method of Zinn-Justin and Brézin[12] straightforwardly as it stands, the simple argument of estimating the relevant contribution to the path integral [analogous to Eqs. (2.75)–

(2.80)] still works. Using again $\Psi = \Psi_0 \exp(-z/\xi)$ as a trial function in the path integral, the three terms resulting from the bracket of Eq. (2.100) make contributions of the order of

$$L_\perp^{d-1}\Psi_0^2/\xi^3, \qquad L_\perp^{d-1} t \Psi_0^2 \xi, \qquad L_\perp^{d-1} u_0 \Psi_0^4 \xi \tag{2.101}$$

For T sufficiently above T_c the first two terms dominate, and putting them equal to each other and equal to unity yields

$$L_\perp^{d-1}\Psi_0^2/\xi^3 = L_\perp^{d-1} t\Psi_0^2 \xi = 1, \qquad \xi = t^{-1/4}, \qquad \Psi_0^2 = t^{-3/4}/L_\perp^{d-1} \tag{2.102}$$

Since again $A(L_\perp, T) \propto \Psi_0^2$, we conclude that $k_B T\chi \propto A(L_\perp, T)$ $L_\perp^{d-1}\xi \propto t^{-1}$, as expected. Again the more interesting case is $t = 0$, where

$$L_\perp^{d-1}\Psi_0^2/\xi_0^3 = L_\perp^{d-1} u_0 \Psi_0^4 \xi_0 = 1, \qquad \Psi_0^2 = 1/u_0 \xi_0^4 \tag{2.103}$$

and

$$\xi_0 = u_0^{-1/7} L_\perp^{(d-1)/7} = u_0^{-1/7} L_\perp L_\perp^{(d-8)/7} \tag{2.104}$$

It is clear that $H_{\text{eff}}(\Psi)$ can be rescaled ($\Psi \to \Psi'$, $z \to z'$) in much the same way as done in Eq. (2.84), and hence we derive for $\xi(L_\perp, T)$ the scaling behavior

$$\xi(L_\perp, T) \simeq t^{-1/4} \tilde{\xi}\left(\frac{L_\perp}{t^{-1/4}} L_\perp^{(d-8)/7}\right) \tag{2.105}$$

with the scaling function $\tilde{\xi}(\zeta \to 0) \propto \zeta$ in order that Eq. (2.105) reduces to Eq. (2.104) as $t \to 0$. The formulas analogous to Eqs. (2.86)–(2.89) become, noting $\Psi_0^2 \propto u_0^{-3/7} L_\perp^{-4(d-1)/7}$ and using Eqs. (2.76) and (2.77) with $A(L_\perp, T) \propto \Psi_0^2$,

$$k_B T_c \chi(T_c) \propto L_\perp^{4(d-1)/7} = L_\perp^{d/2} L_\perp^{(d-8)/14}, \qquad L_{\|} \to \infty \tag{2.106}$$

$$\langle |\Psi| \rangle_{T_c} \propto L_\perp^{-3(d-1)/14} L_{\|}^{-1/2} = (L_\perp^3 L_{\|})^{-1/2} L_\perp^{-3(d-8)/14}, \qquad L_{\|} \to \infty \tag{2.107}$$

Note, however, that Eqs. (2.106) and (2.107) are only assumed to hold if $d > d_{\text{IL}}^* = 8$, while Eqs. (2.86) and (2.87) are thought to hold for $d > d^* = 4$.

Since for $L_{\|}$ finite $L_{\|}$ scales with $\xi(L_\perp, T_c) \equiv \xi_0 \propto L_\perp^{(d-1)/7}$, we find relations analogous to Eqs. (2.88) and (2.89),

$$k_B T_c \chi(T_c) \simeq L_\perp^{d/2} L_\perp^{(d-8)/14} \tilde{\chi}\left(\frac{L_\perp}{L_{\|}} L_\perp^{(d-8)/7}\right) \tag{2.108}$$

$$\langle |\Psi| \rangle_{T_c} \simeq (L_\perp^3 L_{\|})^{-1/2} L_\perp^{-3(d-8)/14} \tilde{\Psi}\left(\frac{L_\perp}{L_{\|}} L_\perp^{(d-8)/7}\right) \tag{2.109}$$

With $\tilde{\chi}(\zeta \to \infty) \to \zeta^{-1/2}$, $\tilde{\Psi}(\zeta \to \infty) \to \zeta^{-1/4}$ the results for the homogeneous case [Eqs. (2.64) and (2.65)] are recovered. The smooth crossover between Eqs. (2.106), (2.107), and Eqs. (2.64), (2.65) now occurs for

$$L_{\|} \propto L_{\perp} L_{\perp}^{(d-8)/7} = L_{\perp}^{(d-1)/7} \tag{2.110}$$

while for ordinary critical phenomena the crossover between Eqs. (2.64), (2.65) and Eqs. (2.86), (2.87) occurs for

$$L_{\|} \propto L_{\perp} L_{\perp}^{(d-4)/3} = L_{\perp}^{(d-1)/3} \tag{2.111}$$

2.8. Uniaxial Lifshitz Points

For a uniaxial Lifshitz point,[21-23] there exists a special direction (which we take the z direction) for which the coefficient of the term $(\partial\Psi/\partial z)^2$ in the gradient energy vanishes, while the coefficients of all other terms, $(\partial\Psi/\partial x_j)^2$, $j = 1, ..., x_{d-1}$, stay nonzero. Thus, an appropriate effective Hamiltonian is, instead of Eq. (2.99),

$$H_{\text{eff}}\{\Psi\} = \int d\mathbf{r} \left[\frac{1}{2} \sum_{j=1}^{d-1} \left(\frac{\partial \Psi}{\partial x_j} \right)^2 + \frac{1}{2} \left(\frac{\partial^2 \Psi}{\partial z^2} \right)^2 + \frac{1}{2} t\Psi^2 + \frac{u_0}{4!} \Psi^4 \right] \tag{2.112}$$

For simplicity, a constant of order unity in front of the term $(\partial^2\Psi/\partial z^2)^2$ has been suppressed. Computing the wavevector-dependent susceptibility $\chi(\mathbf{q})$ in mean-field theory from Eq. (2.112), one finds that[21]

$$[\chi(\mathbf{q})]^{-1} \propto t(1 + \mathbf{q}_{\perp}^2 \xi_{\perp}^2 + q_{\|}^4 \xi_{\|}^4) \tag{2.113}$$

where $\mathbf{q} \equiv (\mathbf{q}_{\perp}, q_{\|})$ and $q_{\|}$ is the component in the z direction, and the correlation lengths $\xi_{\|}, \xi_{\perp}$ in the longitudinal and transverse directions behave as

$$\xi_{\|} \propto t^{-\nu_{\|}}, \qquad \xi_{\perp} \propto t^{-\nu_{\perp}}, \qquad \nu_{\|} = 1/4, \qquad \nu_{\perp} = 1/2 \tag{2.114}$$

in mean-field theory. The generalized hyperscaling relation (1.10) for this uniaxially anisotropic situation holds[21] for $d \leqslant d^* = 4.5$ but is invalid for $d > d^*$, the situation to be considered here.

Due to the anisotropy of the correlation lengths $\xi_{\|}, \xi_{\perp}$, one can no longer conclude that the treatment of Section 2.5 holds for $L_{\|}, L_{\perp}$ being of the same order. Let us therefore first consider the situation where $L_{\|}$ is sufficiently large that nonuniformities in the z direction need to be considered, while in all other directions Ψ is taken as uniform, and identify later the inequality relating $L_{\|}$ and $L_{\perp}$ that must be satisfied in order that this assumption holds. In any case, in this situation the effective Hamiltonian once more is given by Eq. (2.100), and the arguments described in Eqs. (2.101)–(2.110) go through as they stand, the only difference being that they now not only hold for $d > 8$ but for $d > d^* = 4.5$. This suggests that we should rewrite the exponent $y = (d-8)/7$ in the relation $L_{\|} \propto L_{\perp}^{1+y}$ [Eq. (2.110)] for the linear dimension $L_{\|}$ where the smooth crossover occurs between the uniform behavior of the finite system [Eqs. (2.64) and (2.65)] and the nonuniform behavior in the z direction [Eqs. (2.106) and (2.107)]: we do this by relating y to the marginal dimension $d^* = 4.5$ as

$$y = \frac{d-d^*}{7} - \frac{1}{2}, \qquad L_{\|} \propto L_{\perp}^{1/2} L_{\perp}^{(d-d^*)/7}, \qquad d > d^* = 4.5 \quad (2.115)$$

Also, Eqs. (2.108) and (2.109) can be rewritten in a form which emphasizes the smooth crossover at $L_{\|}$ as given by Eq. (2.115) more clearly,

$$k_B T_c \chi(T_c) \simeq L_{\perp}^{d/2 - 1/4} L_{\perp}^{(d-d^*)/14} \tilde{\chi}\left(\frac{L_{\perp}^{1/2}}{L_{\|}} L_{\perp}^{(d-d^*)/7}\right) \qquad (2.116)$$

$$\langle |\Psi| \rangle_{T_c} \simeq L_{\perp}^{-3/4} L_{\|}^{-1/2} L_{\perp}^{-3(d-d^*)/14} \tilde{\Psi}\left(\frac{L_{\perp}^{1/2}}{L_{\|}} L_{\perp}^{(d-d^*)/7}\right) \qquad (2.117)$$

It is reassuring to note that at $d = d^* = 4.5$, neglecting the expected logarithmic correction, we obtain from Eq. (2.116) precisely the same behavior as from Eq. (2.34), rewritten in the form

$$k_B T_c \chi(T_c) \simeq L_{\|}^{\gamma/\nu_{\|}} \tilde{\chi}'\left(\frac{L_{\|}}{L_{\perp}^{\nu_{\|}/\nu_{\perp}}}\right) = L_{\perp}^{\gamma/\nu_{\perp}} \tilde{\chi}\left(\frac{L_{\perp}^{\nu_{\|}/\nu_{\perp}}}{L_{\|}}\right) \qquad (2.118)$$

if we use $\gamma = 1$, $\nu_{\|} = 1/4$, $\nu_{\perp} = 1/2$ [Eqs. (2.113) and (2.114)], and then $k_B T_c \chi(T_c) \simeq L_{\perp}^2 \tilde{\chi}(L_{\perp}^{1/2}/L_{\|})$. Assuming one more time that

$$\langle |\Psi| \rangle_{T_c} \approx \langle \Psi^2 \rangle_{T_c}^{1/2} = [k_B T_c \chi(T_c)/L_{\perp}^{d-1} L_{\|}]^{1/2}$$

yields then

$$\langle |\Psi| \rangle_{T_c} \simeq L_{\perp}^{-3/4} L_{\|}^{-1/2} [\tilde{\chi}(L_{\perp}^{1/2}/L_{\|})]^{1/2}$$

i.e., a result having the structure of Eq. (2.117). Thus, the treatment of Section 2.4, based on generalized hyperscaling and believed to be valid for $d < d^*$, and the present treatment based on a generalized mean-field approach which should hold for $d > d^*$, yield an identical scaling structure precisely for $d = d^*$. Of course, this is a necessary consistency check of the present theory. We also emphasize the fact that for such anisotropic systems, where $\nu_{\|} \neq \nu_{\perp}$, a finite system with all linear dimensions $L_{\|}$, $L_{\perp}$ of the same order exhibits at T_c a nonuniformly ordered state; in the present case of uniaxial Lifshitz points, the nonuniformity does occur in the z direction (since $\nu_{||} < \nu_{\perp}$), and the correlation function in the z direction decays exponentially with a correlation length $\xi_0 \propto L_{\perp}^{\nu_{||}/\nu_{\perp}}$ for $d < d^*$ and with $\xi_0 \propto L_{\perp}^{1/2} L_{\perp}^{(d-d^*)/7}$ for $d > d^*$.

We now wish to consider a geometry which is elongated in one transverse direction (the "thin film" geometry where all transverse directions becomes very large is outside of consideration here). Thus, the geometry is $L'_{\perp} \times L_{\perp}^{d-2} \times L_{||}$ with $L'_{\perp} \gg L_{\perp}$, $L'_{\perp}$ being the linear dimension in the x direction. We choose $L_{||}$ at most of the size as defined in Eq. (2.115): then the only relevant inhomogeneities occur in the x direction, while all other inhomogeneities can be integrated out perturbatively. Then the effective Hamiltonian becomes

$$H_{\text{eff}}\{\Psi\} = L_{\perp}^{d-2} L_{||} \int_0^{L'_{\perp}} dx \left[\frac{1}{2}\left(\frac{d\Psi}{dx}\right)^2 + \frac{1}{2} t\Psi^2 + \frac{u_0}{4!}\Psi^4\right] \tag{2.119}$$

We immediately recognize that this problem is equivalent to Eq. (2.74), only z is relabeled as x, $L_{||}$ is relabeled as $L'_{\perp}$, and $L_{\perp}^{d-1}$ is replaced by $L_{\perp}^{d-2} L_{||}$. Thus we conclude that Eqs. (2.81) and (2.82) are replaced by

$$\xi_0 \simeq u_0^{-1/3} (L_{\perp}^{d-2} L_{||})^{1/3} \tag{2.120}$$

$$\xi(L_{\perp}, L_{||}, t) \simeq t^{-1/2} \tilde{\xi}\left(\frac{(L_{\perp}^{d-2} L_{||})^{1/3}}{t^{-1/2}}\right) \tag{2.121}$$

Equations (2.88) and (2.89) get replaced by

$$k_B T_c \chi(T_c) \simeq (L_{\perp}^{d-2} L_{||})^{2/3} \tilde{\chi}\left(\frac{(L_{\perp}^{d-2} L_{||})^{1/3}}{L'_{\perp}}\right) \tag{2.122}$$

$$\langle |\Psi| \rangle_{T_c} \simeq (L'_{\perp})^{-1/2} (L_{\perp}^{d-2} L_{||})^{-1/6} \tilde{\Psi}\left(\frac{(L_{\perp}^{d-2} L_{||})^{1/3}}{L'_{\perp}}\right) \tag{2.123}$$

It needs to be emphasized that this holds only if $L_{||}$ is small enough. The maximum possible choice for $L_{||}$ is given by Eq. (2.115); then Eqs. (2.122) and (2.133) can be rewritten as

$$k_B T_c \chi(T_c) \simeq (L_{\perp}^{d-3/2} L_{\perp}^{(d-d^*)/7})^{2/3} \tilde{\chi}\left(\frac{(L_{\perp}^{d-3/2} L_{\perp}^{(d-d^*)/7})^{1/3}}{L'_{\perp}}\right) \tag{2.124}$$

$$\langle |\Psi| \rangle_{T_c} \simeq (L'_{\perp})^{-1/2} (L_{\perp}^{d-3/2} L_{\perp}^{(d-d^*)/7})^{-1/6} \tilde{\Psi}\left(\frac{(L_{\perp}^{d-3/2} L_{\perp}^{(d-d^*)/7})^{1/3}}{L'_{\perp}}\right) \tag{2.125}$$

It is interesting to note that in general ($d > d^*$) $L'_\perp$ does not simply scale with $L_\perp$, although the decay of the bulk correlation function is described by the same exponent $\nu_\perp = 1/2$ in all transverse directions. Putting again $d = d^* = 4.5$, however, we do recover simple scaling laws,

$$k_B T_c \chi(T_c) \simeq L_\perp^2 \tilde{\chi}\left(\frac{L_\perp}{L'_\perp}\right), \qquad d = d^* = 4.5 \tag{2.126}$$

$$\langle|\Psi|\rangle_{T_c} \simeq L'^{-1/2}_\perp L_\perp^{-1/2} \tilde{\Psi}\left(\frac{L_\perp}{L'_\perp}\right), \qquad d = d^* = 4.5 \tag{2.127}$$

which are consistent with the analogous expressions for $d \leqslant d^*$,

$$k_B T_c \chi(T_c) \simeq L_\perp^{\gamma/\nu_\perp} \tilde{\chi}\left(\frac{L_\perp}{L'_\perp}\right) \tag{2.128}$$

$$\langle|\Psi|\rangle_{T_c} \simeq L_\perp^{-\beta/\nu_\perp} \tilde{\Psi}\left(\frac{L_\perp}{L'_\perp}\right) = L_\perp^{-\beta/2\nu_\perp} L_\perp^{-\beta/2\nu_\perp} \tilde{\Psi}'\left(\frac{L_\perp}{L'_\perp}\right) \tag{2.129}$$

2.9. Finite-Size Scaling for the Driven Kawasaki Model in its Field-Theoretic Version

This subsection relies on the basic assumption that the treatment of the previous sections, dealing with anisotropic phase transitions in thermal equilibrium, can be carried over to an anisotropic nonequilibrium phase transition as well. Thus, the results in this subsection are necessarily speculative. However, no other approach exists for the driven Kawasaki model and it is doubtful whether a renormalization group theory of finite-size effects is feasible.[44]

We start by considering once more the geometry $L'_\perp \times L_\perp^{d-2} \times L_\|$ as just treated, with $L'_\perp \gg L_\perp$ but $L_\|$ not too large to ensure that the only relevant inhomogeneities again occur in the x direction where the linear dimension is $L'_\perp$. The maximum choice for $L_\|$, of course, is not known at this point, but will be derived below.

Now the treatments of Leung and Cardy[35] and Janssen and Schmittmann[34] are invoked to conclude that the driven Kawasaki model for $d > 2$ should behave mean-field-like, the exponents being given in Eq. (1.11). But then our implication is that Eqs. (2.119)–(2.123) must describe this model, too!

It is instructive to consider the limiting situation at $d = 2$ where Eq. (1.11) would be compatible with the generalized hyperscaling relation (1.10). Of course, for $d = 2$, a single transverse direction $L'_\perp$ remains, and hence, using Eqs. (2.122) and (2.123) for $d=2$, $L_\perp$ cancels out. Relabeling then $L'_\perp$ as $L_\perp$, we get

$$k_B T_c \chi(T_c) \simeq L_{\|}^{2/3} \tilde{\chi}(L_{\|}^{1/3}/L_\perp) \tag{2.130}$$

$$\langle |\Psi| \rangle_{T_c} \simeq L_\perp^{-1/2} L_{\|}^{-1/6} \tilde{\Psi}(L_{\|}^{1/3}/L_\perp) \tag{2.131}$$

Again this is seen to be consistent with the scaling result (2.34) if we recall $\eta_{\|}=4/3$, $\nu_{\|}=3/2$ for $d=2$ in this model.

Now we have to finally consider the geometry $L_\perp^{d-1} L_{\|}$ with $L_{\|} \to \infty$. In this case the construction of $H_{\text{eff}}\{\Psi\}$ is a rather delicate matter: on one hand, it should reproduce the result that for $t<0$ the equation of state must have the simple Landau[34] form for all $d>2$. This implies that the homogeneous part of the integrand of H_{eff} must still be given simply by $\frac{1}{2}t\Psi^2+(u_0/4!)\Psi^4$, as in Eq. (2.112). On the other hand, the gradient energy in the z direction can no longer be $\frac{1}{2}(d\Psi/dz)^2$, since that then invariably would imply $\nu_{\|}=1/2$, and we rather must get $\nu_{\|}=1+(5-d)/6$; see Eq. (2.11). This consideration shows that a *simple* Landau theory cannot work as an integrand of the functional as it was used in Eqs. (2.59), (2.74), (2.100) and (2.112); we must work with a generalized Landau theory[43] where the gradient term is singular. This difficulty reflects the fact[34,35] that the fixed point for $d<5$ which yields $\nu_{\|}$ is not the Gaussian fixed point but another nontrivial fixed point, explicitly calculated in refs. 34 and 35. In this respect, the situation clearly differs from the Lifshitz-point problems considered above.

Thus, we make the *speculative* proposal to work with the effective functional

$$H_{\text{eff}}\{\Psi\} = L_\perp^{d-1} \int_0^{L_{\|}} dz \left[\frac{1}{2}\left(\frac{d^\kappa \Psi}{d^\kappa z}\right)^2 + \frac{1}{2} t\Psi^2 + \frac{u_0}{4!}\Psi^4\right] \tag{2.132}$$

where now an exponent $\kappa<1$ makes the derivative term singular, while at the Lifshitz point we had $\kappa=2$ and for ordinary critical points $\kappa=1$. The value of κ is specified below.

Working once more with the trial function $\Psi(z)=\Psi_0\exp(-z/\xi)$ consistent with an exponential falloff of correlations, which is always expected to occur with a geometry $L_{\|}\to\infty$, $L_\perp$ finite, we estimate the terms in Eq. (2.132) analogously to Eqs. (2.78), (2.101) as

$$L_\perp^{d-1}\Psi_0^2\xi^{1-2\kappa}, \qquad L_\perp^{d-1} t\Psi_0^2\xi, \qquad L_\perp^{d-1} u_0 \Psi_0^4 \xi \tag{2.133}$$

For T sufficiently above T_c the first two terms dominate: putting them equal to each other and equal to unity yields

$$L_\perp^{d-1}\Psi_0^2\xi^{1-2\kappa} = L_\perp^{d-1} t\Psi_0^2 \xi = 1, \qquad \xi = t^{-1/(2\kappa)} \tag{2.134}$$

Requesting that $\xi \propto t^{-1-\varepsilon/6}$ $(\varepsilon = 5-d)$ thus yields

$$\kappa = \frac{1}{2(1+\varepsilon/6)} \tag{2.135}$$

Since $\Psi_0^2 = L_\perp^{-(d-1)}/t\xi$ and [Eq. (2.76)] $k_B T\chi \approx \Psi_0^2 L_\perp^{d-1}\xi = t^{-1}$, this result reproduces the desired critical behavior for $t > 0$.

At $t = 0$ we now equate the first and the third terms to each other and to unity,

$$L_\perp^{d-1}\Psi_0^2\xi_0^{1-2\kappa} = L_\perp^{d-1}u_0\Psi_0^4\xi_0 = 1, \qquad \Psi_0^2 = \xi_0^{-2\kappa}/u_0 \tag{2.136}$$

and hence

$$\begin{aligned}\xi_0 &= u_0^{-1/(4\kappa-1)}L_\perp^{(d-1)/(4\kappa-1)} = u_0^{-(1+\varepsilon/6)/(1-\varepsilon/6)}L_\perp^{(d-1)[(1+\varepsilon/6)/(1-\varepsilon/6)]} \\ &\propto L_\perp^3 L_\perp^{-(d-2)[(d-7)/(d+1)]}\end{aligned} \tag{2.137}$$

Thus, the result analogous to Eq. (2.82) is

$$\xi(L_\perp, T) \simeq t^{-[1+(5-d)/6]}\, g\left(\frac{L_\perp^3 L_\perp^{-(d-2)(d-7)/(d+1)}}{t^{-[1+(5-d)/6]}}\right) \tag{2.138}$$

At $d = 2$, we again find $\xi_0 \propto L_\perp^3$, consistent with the findings of Eqs. (2.130) and (2.131). For $d = 3$, 4, and 5, the respective results would be, however, $\xi_0 \propto L_\perp^4$, $L_\perp^{21/5}$, and $L_\perp^4$.

Using ξ_0 in Eq. (2.136) to estimate Ψ_0^2 as

$$\Psi_0^2 \propto L_\perp^{-(d-1)/2}u_0^{-1/2}\xi_0^{-1/2} \propto L_\perp^{-(d-1)[1+(1+\varepsilon/6)/(1-\varepsilon/6)]/2} \tag{2.139}$$

we obtain $\chi(T_c)$ from an expression analogous to Eq. (2.76),

$$\begin{aligned}k_B T_c\chi(T_c) &\propto L_\perp^{(d-1)[1+(1+\varepsilon/6)/(1-\varepsilon/6)]/2} \\ &= L_\perp^{(d+2)/2}L_\perp^{-(d-2)(d-7)/2(d+1)}, \qquad L_\| \to \infty\end{aligned} \tag{2.140}$$

and similarly, using Eq. (2.77),

$$\langle|\Psi|\rangle_{T_c} \propto L_\perp^{-(d-4)/4}L_\|^{-1/2}L_\perp^{-(d-2)(d-7)/4(d+1)}, \qquad L_\| \to \infty \tag{2.141}$$

Since $L_\|$ scales with ξ_0 as written in Eq. (2.137), we may further conclude that

$$k_B T_c\chi(T_c) \simeq L_\perp^{(d+2)/2}L_\perp^{-(d-2)(d-7)/2(d+1)}\tilde{\chi}\left(\frac{L_\perp^3}{L_\|}L_\perp^{-(d-2)(d-7)/(d+1)}\right) \tag{2.142}$$

$$\langle|\Psi|\rangle_{T_c} \simeq L_\|^{-1/2}L_\perp^{-(d-4)/4}L_\perp^{-(d-2)(d-7)/2(d+1)}\tilde{\Psi}\left(\frac{L_\perp^3}{L_\|}L_\perp^{-(d-2)(d-7)/(d+1)}\right) \tag{2.143}$$

At $d=2$ we recover the result $k_B T_c \chi(T_c) \simeq L_\perp^2 \chi(L_\perp^3/L_{||})$ compatible with Eq. (2.130), and

$$\langle |\Psi| \rangle_{T_c} \simeq L_{||}^{-1/2} L_\perp^{1/2} \tilde{\Psi}(L_\perp^3/L_{||}) = L_{||}^{-1/3} \tilde{\Psi}'(L_\perp^3/L_{||})$$

consistent with Eq. (2.131).

Again we wish to consider the crossover from the strip geometry to the fully homogeneous situation, as discussed in Section 2.5. This crossover should occur for $\xi_0 \approx L_{||}$, i.e.,

$$L_{||} \propto L_\perp^3 L_\perp^{-(d-2)(d-7)/(d+1)} \tag{2.144}$$

Using Eq. (2.65) for $\chi(T_c)$ and $L_{||}$ as given by Eq. (2.144) yields

$$k_B T_c \chi(T_c) \propto L_\perp^{(d-1)/2} L_\perp^{3/2} L_\perp^{-(d-2)(d-7)/2(d+1)} \tag{2.145}$$

which agrees with Eq. (2.140), and using Eq. (2.64) for $\langle |\Psi| \rangle_{T_c}$ together with Eq. (2.144) yields

$$\langle |\Psi| \rangle_{T_c} \propto L_\perp^{-(d-1)/4} L_\perp^{-3/4} L_\perp^{(d-2)(d-7)/4(d+1)} \tag{2.146}$$

which is the same result as Eq. (2.141) combined with Eq. (2.144). Of course, the smoothness of all these crossovers at arbitrary d in the range $2 \leqslant d \leqslant 5$ is a necessary, but not sufficient, condition for the correctness of the present theory.

3. MONTE CARLO STUDY OF SHAPE EFFECTS ON FINITE-SIZE SCALING: RECTANGULAR $L_{||} \times L_\perp$ SUBSYSTEMS IN THE TWO-DIMENSIONAL ISING MODEL

In this section we describe some Monte Carlo simulations which test some of the concepts developed in Section 2.2: this is the simplest case, where the correlation function decay is still governed by the same length in the two lattice directions (x, y), and thus it is only a system shape effect that is under study.

This finite-size scaling analysis of subsystems of a very large square having sides $L_{max} \times L_{max}$ and periodic boundary conditions, and the subsystems having the geometry of $L_{||} \times L_\perp$ with both $L_{||} \ll L_{max}$ and $L_\perp \ll L_{max}$, is attractive for a variety of reasons: a single simulation run of the one large system is sufficient to yield the complete finite-size information; in addition, there is evidence that, at least for $L_{||} = L_\perp$ the corrections to finite-size scaling seem to be rather small.[5] If instead we would consider fully finite $L_{||} \times L_\perp$ blocks with periodic boundary conditions, each geometry $(L_{||}, L_\perp)$ requires a separate simulation run.

On the other hand, the finite-size scaling analysis of subsystems also has distinct disadvantages, from the computational point of view: if we were to perform a Monte Carlo simulation applying standard single spin-flip algorithms,[45] the extension[46] of finite-size scaling to dynamic critical phenomena[47] implies that the relaxation time τ at T_c scales with the linear dimensions $L_{\|}$, $L_{\perp}$, and $L_{\max}$ as follows:

$$\tau \propto L_{\max}^{z} \qquad \text{(subsystem geometry)} \tag{3.1}$$

$$\tau \propto (L_{\|} L_{\perp})^{z/2}\, \tilde{\tau}(L_{\|}/L_{\perp}) \qquad \text{(finite block geometry)} \tag{3.2}$$

where z is the dynamic critical exponent of the kinetic Ising model without conservation law[47] ($z \approx 2.13$[48]). The scaling function $\tilde{\tau}(\zeta)$ is a constant when the argument is of order unity and $\tilde{\tau}(\zeta \to \infty) = \tilde{\tau}(1/\zeta \to 0) \propto \zeta^{-z/2}$. Since we must have $L_{\max} \gg L_{\|}$, $L_{\perp} \gg 1$, it is clear that a meaningful finite-size study of subsystem shape effects requires an extreme computational

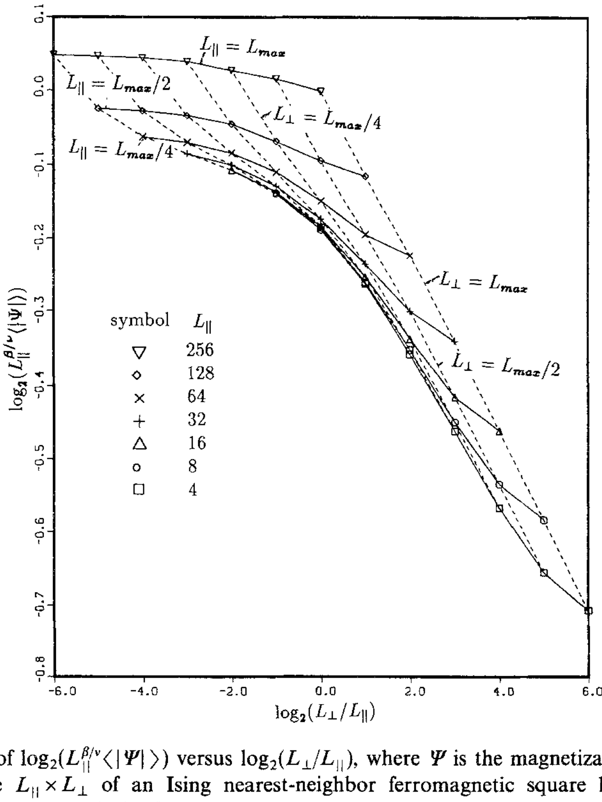

Fig. 1. Plot of $\log_2(L_{\|}^{\beta/\nu}\langle|\Psi|\rangle)$ versus $\log_2(L_{\perp}/L_{\|})$, where Ψ is the magnetization in a subsystem of size $L_{\|} \times L_{\perp}$ of an Ising nearest-neighbor ferromagnetic square lattice of size $L_{\max} \times L_{\max}$ with $L_{\max} = 256$. The data refer to $T = T_c$. Curves are only drawn to guide the eye. Standard Ising exponents ($\beta = 1/8$, $\nu = 1$) are used.

effort, since the observation time must by far exceed the relaxation times in order to obtain meaningful results.

Such a study is nevertheless feasible, however, applying the Monte Carlo algorithm,[49] where rather than flipping a single spin, in each step an arbitrarily large cluster of spins can be overturned: this algorithm reduces the "inefficiency" of the Monte Carlo method at T_c by strongly reducing the value of the exponent z, namely $z \approx 0.35$.[49] Thus we shall apply this algorithm in the following.

First we describe results for $L_{\max} = 256$ (Figs. 1 and 2) for the two-dimensional Ising model right at T_c ($J/k_B T_c \approx 0.4407$[50]). This size $L_{\max}$ is large but not very large—it is about the maximum size that could be studied even with the single-spin flip algorithm with reasonable effort. Since it is not clear *a priori* how restrictively the conditions $L_{\|} \ll L_{\max}$, $L_{\perp} \ll L_{\max}$ should be interpreted, we have ignored them in Figs. 1 and 2. The penalty for this procedure is drastic deviation from scaling: the data with $L_{\|} = L_{\max}$

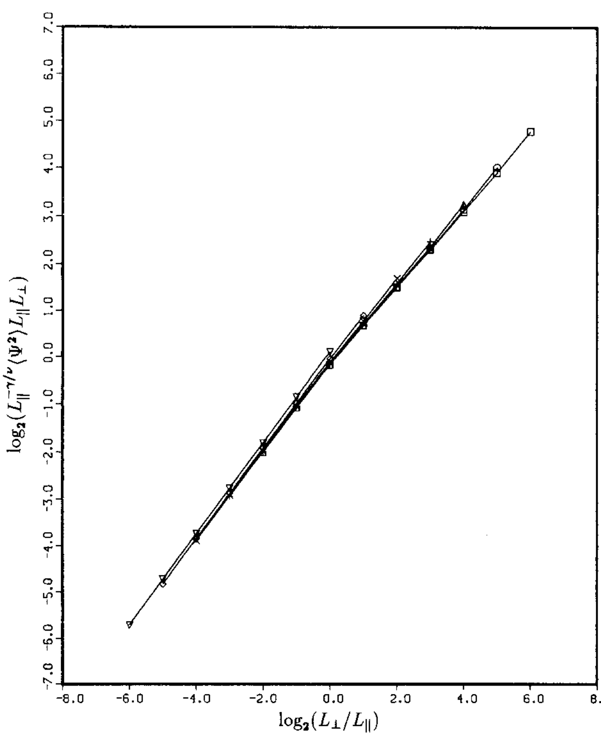

Fig. 2. Plot of $\log_2(L_{\|}^{-\gamma/\nu}\langle\Psi^2\rangle L_{\|}L_{\perp})$ versus $\log_2(L_{\perp}/L_{\|})$ for the same calculation as shown in Fig. 1. Curves are only drawn to guide the eye. Standard Ising model exponents ($\gamma = 7/4$, $\nu = 1$) are used.

or $L_\perp = L_{\max}$ clearly are far off from the scaling function, and also the data with $L_{||} = L_{\max}/2$ or $L_\perp = L_{\max}/2$ fail to scale. The curves which satisfy both $L_{||} < L_{\max}/4$ and $L_\perp < L_{\max}/4$ start converging against an envelope, which is the expected scaling function. On the other hand, since we want $L_{||} \gg 1$, $L_{\max} \gg 1$, it certainly makes no sense to include data smaller than $L_{||} = 4$ or $L_\perp = 4$ in such a plot: thus, only a few combinations of $(L_{||}, L_\perp)$ are close to the scaling limit if $L_{\max} = 256$. Thus, with data as presented in Figs. 1 and 2 a conclusive test of the scaling ideas presented in Section 2.2 cannot be performed.

Thus, we have undertaken a somewhat larger computational effort, choosing $L_{\max} = 1024$, which took about 5 h CPU time at a CYBER 205 supercomputer,[5] To our knowledge, this is the largest two-dimensional

[5] Actually the limitation which prevented us in choosing $L_{\max}$ still larger is not the CPU time but the storage requirement for the algorithm of ref. 49.

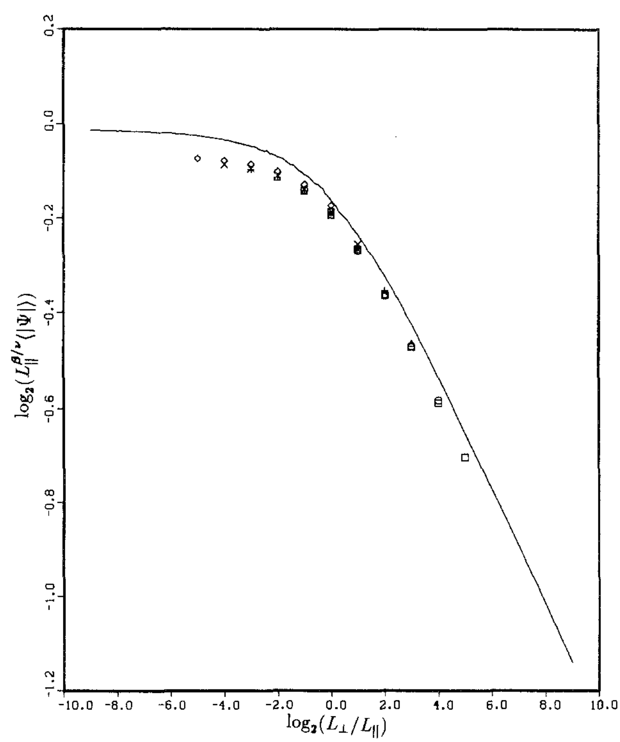

Fig. 3. Log_2–$\log_2$ plot of $L_{||}^{\beta/\nu}\langle|\Psi|\rangle$ versus $L_\perp/L_{||}$, using subsystems $L_{||} \times L_\perp$ of a system of size $L_{\max} \times L_{\max}$ with $L_{\max} = 1024$. Subsystems included are all combinations $(L_{||}, L_\perp)$ with linear dimensions 4, 8, 16, 32, 64, and 128. The smooth curve is an approximate calculation of the scaling function, as described in the text.

system that ever was equilibrated right at the critical point: availability of the novel algorithm of ref. 49 was absolutely crucial for this effort—even using the superfast vectorizing multispin algorithms currently available[51] for the simulation of Ising models would take about 100 times the effort for this $L_{\max}$.

Figures 3–5 show that with this large size we can get convincing verification of the shape-dependent finite-size scaling at T_c [Eqs. (2.7) and (2.13)]. Motivated by the experience demonstrated in Figs. 1 and 2, we have included only data with both $L_{\parallel}$ and $L_{\perp} \leqslant L_{\max}/8$ in these figures. If we include all sizes, deviations from scaling are again clearly visible and the scaling only appears as an envelope of parts of these curves, as demonstrated in Fig. 6 for the fourth-order cumulant g [Eq. (2.12)].

Since the correlation functions of the two-dimensional nearest neighbor Ising model are known exactly,[40,52] one can evaluate Eq. (2.4) to derive the scaling function $\tilde{\chi}(L_{\parallel}/L_{\perp})$ in Eq. (2.7) numerically. Instead

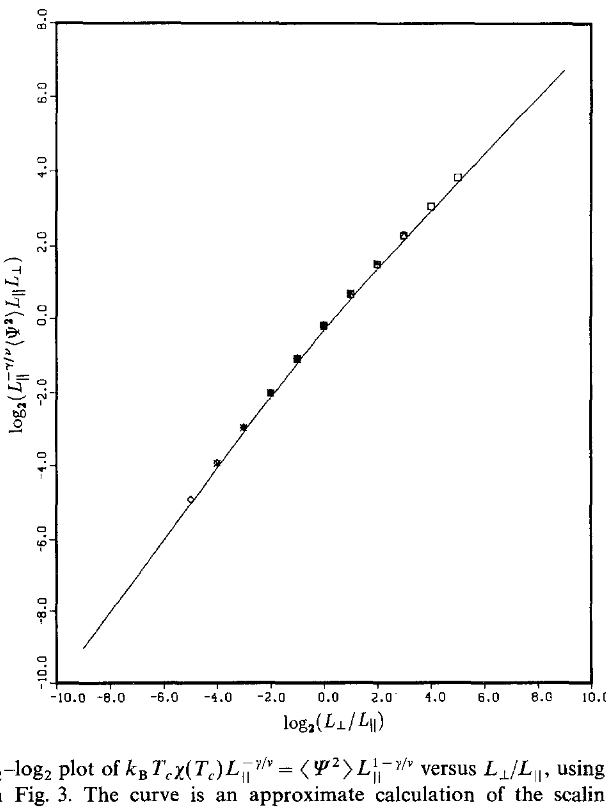

Fig. 4. Log_2–$\log_2$ plot of $k_B T_c \chi(T_c) L_{\parallel}^{-\gamma/\nu} = \langle \Psi^2 \rangle L_{\parallel}^{1-\gamma/\nu}$ versus $L_{\perp}/L_{\parallel}$, using the same subsystems as in Fig. 3. The curve is an approximate calculation of the scaling function, as described in the text.

working with the complete expansion derived in refs. 40 and 52, we have used the leading-order term only

$$\langle \Psi(0,0)\,\Psi(x,y)\rangle_{T_c} \approx 0.645(x^2+y^2)^{-1/8} \tag{3.3}$$

and neglected corrections which are of order[40] $(x^2+y^2)^{-5/8}$. The asymptotic form of the scaling function is $\tilde{\chi}(x) \propto x^{3/4}$ as $x \to \infty$, and $\tilde{\chi}(x) \propto x$ as $x \to 0$. It is seen that the resulting approximate scaling function is in good agreement with the simulation data (Fig. 4). Using then Eqs. (2.9) and (2.10) to estimate $\tilde{\Psi}(L_{\|}/L_{\perp}) \approx [\tilde{\chi}(L_{\|}/L_{\perp})]^{1/2}$, the general trend of the actual scaling function $\tilde{\Psi}$ is also correctly predicted, although quantitative agreement no longer can be obtained, as expected.

An interesting feature of the cumulant $g(L_{\|}/L_{\perp})$ (Fig. 5) is the symmetry of the curve around the line $\log_2(L_{\|}/L_{\perp})=0$. This symmetry just reflects the symmetry $g(L_{\|}/L_{\perp}) = g(L_{\perp}/L_{\|})$, of course, since $\log(L_{\|}/L_{\perp}) = -\log(L_{\perp}/L_{\|})$. This symmetry property was not imposed in the actual

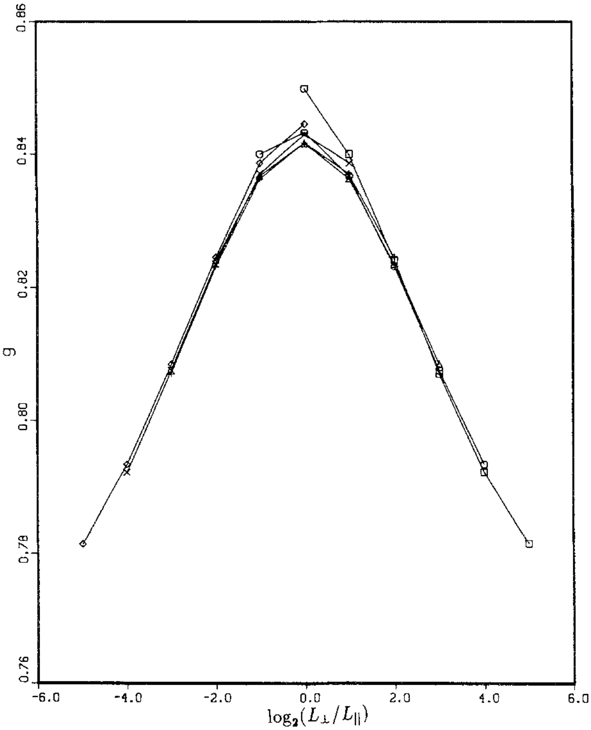

Fig. 5. Plot of g versus $\log_2(L_\perp/L_\|)$, with $L_\| = 128$ (diamonds), $L_\| = 64$ (crosses), $L_\| = 32$ (pluses), $L_\| = 16$ (triangles), $L_\| = 8$ (circles), and $L_\| = 4$ (squares). Curves are only guides to the eye. All data refer to the same calculation as shown in Figs. 3 and 4.

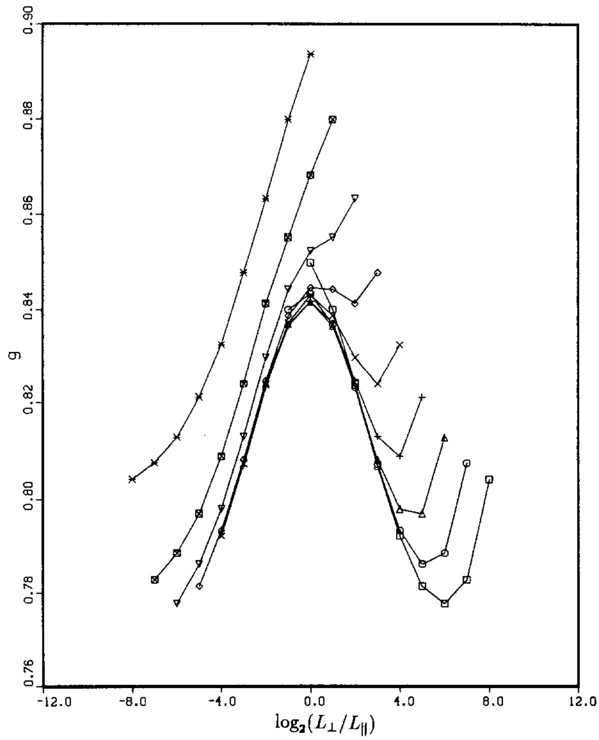

Fig. 6. Same as Fig. 5 but including now all values of $L_\perp$ up to $L_\perp = L_{max}$, as well as three additional values for $L_\parallel$: $L_\parallel = 256$ (standing triangles), $L_\parallel = 512$ (square with crosses), and $L_\parallel = 1024$ (asterisk).

direct calculation of g and thus is a sensitive test of the accuracy of the simulation data in Fig. 5 (note that statistical errors of g are larger than the errors of χ and of $\langle |\Psi| \rangle$, respectively). It is also clear that the maximum of g occurs for $L_\parallel = L_\perp$—then the correlations in the system are closest to those of a fully ordered state (for a perfectly aligned spin configuration g attains its maximal possible value $g = 1$).

We end this section with a caveat: including data which are inappropriate (e.g., if $L_\parallel$ and $L_\perp$ are not small in comparison with L_{max}; see Figs. 1 and 2) leads to pronounced deviations from finite-size scaling. However, the present situation is rather clear-cut, since both T_c and the critical exponents are known exactly.[50,52] In a case where both T_c and critical exponents are extracted from the finite-size scaling analysis itself, one might be easily mislead by results such as in Figs. 1 and 2 to choose T_c, β, and ν somewhat in error to obtain an apparently better "data collapsing" on the finite-size scaling plot.

4. CONCLUSIONS

In this paper we have addressed the effect of anisotropy on finite-size scaling at critical points: both anisotropy of shape (e.g., rectangular systems with linear dimensions $L_{\|}$ and $L_{\perp}$ different in the two lattice directions) and anisotropic critical behavior (correlation lengths $\xi_{\|}$ and $\xi_{\perp}$ diverging with different critical exponents $\nu_{\|}$ and $\nu_{\perp}$ in different directions) are considered. The special case where $L_{\|} \neq L_{\perp}$ but correlations behave isotropically is treated also, since it can serve very well as a simple test case for some of the concepts developed in our work.

Since one main motivation of this work is to provide a framework for the analysis of computer simulations of nonequilibrium phase transitions such as the driven Kawasaki model, for which field-theoretic treatments predict that for three dimensions the generalized hyperscaling relation does not hold, we have paid particular attention to finite-size effects without involving hyperscaling. For equilibrium phase transitions, this situation arises for uniaxial Lifshitz points above their marginal dimension $d^* = 4.5$, and hence this is a problem of theoretical interest only; nevertheless, our tentative generalization of the treatment of Brézin and Zinn–Justin to this anisotropic situation is a challenge which calls for a more rigorous treatment of finite-size effects for such anisotropic mean-field problems.

A key idea of our treatment is to ask whether the finite system (which has $d-1$ linear dimensions $L_{\perp}$ and one linear dimension $L_{\|}$) exhibits a uniform order or a nonuniformly ordered state when we cool it down through T_c. It is well known that for $d=2$ and $L_{\|} \to \infty$ the ferromagnetic Ising system is nonuniformly ordered down to $T \to 0$; one obtains a succession of domains of opposite magnetization separated by domain walls [which are at random positions, at an average distance $\xi_{\|}$, consistent with a spin correlation function proportional to $\exp(-z/\xi_{\|})$ for spins separated a distance z along the strip]. In the critical region, the growth of correlations (in an Ising system the correlations above T_c grow uniformly according to the power law $\xi \propto t^{-\nu}$, of course) is limited when ξ becomes of the order of the smaller length $L_{\perp}$, and thus $\xi_{\perp}$ is of order $L_{\perp}$ in the critical region (sufficiently below T_c, $\xi_{\perp}$ varies exponentially with $L_{\perp}$, of course; this situation is outside of consideration here). The fact that $\xi_{\perp} \propto L_{\perp}$ at T_c is the basis for the well-known phenomenological renormalization technique for strips of finite width.

While an isotropic system is uniformly ordered for $L_{\|} = L_{\perp}$, this is not true for systems with anisotropically diverging correlations: if $\nu_{\|} > \nu_{\perp}$, the system will exhibit finite-size rounding if (for $d < d^*$) $L_{\|} \approx \xi_{\|} \propto t^{-\nu_{\|}}$, although then $\xi_{\perp} \propto t^{-\nu_{\perp}}$ is still much smaller than $L_{\perp}$ $(=L_{\|})$. A transition to uniform order is then obtained for linear dimensions $L_{\|}$ and $L_{\perp}$ related as $L_{\|} \propto L_{\perp}^{\nu_{\|}/\nu_{\perp}}$. If $L_{\|}$ is much larger than $L_{\perp}^{\nu_{\|}/\nu_{\perp}}$, the growth of correlation is first limited in the transverse directions and the correlation function in the z direction (calling $L_{\|}$) will exhibit at T_c an exponential decay (with an effective correlation length proportional to $L_{\perp}^{\nu_{\|}/\nu_{\perp}}$); on the other hand, if $L_{\|}$ is much less than $L_{\perp}^{\nu_{\|}/\nu_{\perp}}$, the growth of correlations is first limited in the z direction, and the correlation in the transverse directions will exhibit at T_c an exponential decay, but now with an effective correlation length $L_{\|}^{\nu_{\perp}/\nu_{\|}}$.

Quantitatively similar conclusions hold also for $d > d^*$, where it is no longer the relation $L_{\|} \propto L_{\perp}^{\nu_{\|}/\nu_{\perp}}$ which controls the crossover from uniform to nonuniform order at T_c, however. Instead we predict for uniaxial Lifshitz points a crossover at $L_{\|} \propto L_{\perp}^{1/2} L_{\perp}^{(d-d^*)/7}$ (with $d^* = 4.5$) and for the driven Kawasaki model at $L_{\|} \propto L_{\perp}^{3} L_{\perp}^{-(d-2)(d-7)/(d+1)}$

While our scaling description for shape effects in the standard two-diensional Ising system is in full agreement with previous treatments based on conformal invariance and is easily confirmed by simulations (see Section 3), unfortunately no such Monte Carlo tests are available for the other scaling predictions made in the present work. Such Monte Carlo studies for anisotropic systems are not straightforward: in the case of Lifshitz points, it is already difficult to locate them in the parameter space of simple models, such as the ANNNI model.[53] Moreover, the regime $d > d^*$ is hard to study; one would have to consider 5-dimensional lattices. More work exists for the driven Kawasaki model, particularly for $d = 2$: but in contrast to our description, available preliminary data even might be compatible with $\nu_{\|} = \nu_{\perp}$! In order to resolve this problem, much more extensive Monte Carlo work is necessary. This work will be presented and analyzed in a future publication.

Finally, we mention that anisotropic scaling also occurs for wetting transitions, where an interface between coexisting bulk phases bound to a wall in a semi-infinite system in the "nonwet" phase unbinds at transition and is delocalized deep in the bulk in the "wet" phase (see ref. 54 for a general review). Size effects on such wetting transitions have been discussed by several authors[55–59] and have also been seen in various Monte Carlo simulations,[58,60] but are outside of consideration here.

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NOTE ADDED IN PROOF

After completion of our paper, we learned about the beautiful work of Bhattacharjee and Nagle,[61] who analyzed finite-size scaling for the exactly solved Kasteley model.[62] This model exhibits an exact example for anisotropic finite size scaling of the type considered here in Section 2.3. It has been shown[61] that the specific heat scales like

$$C \simeq L_{\parallel}^{\alpha/\nu_{\parallel}} \tilde{C}(tL_{\parallel}^{1/\nu_{\parallel}},\ L_{\perp}^{1/\nu_{\perp}}/L_{\parallel}^{1/\nu_{\parallel}})$$

where the exponents are known exactly ($\alpha = 1/2$, $\nu_{\parallel} = 1$, $\nu_{\perp} = 1/2$),[61] satisfying the anisotropic hyperscaling relation $\nu_{\parallel} + \nu_{\perp} = 2 - \alpha$, and also the scaling function $\tilde{C}$ is known exactly. It has also been shown that for $L_{\perp} \leqslant 10$, pronounced deviations from finite size scaling occur.[61]

This model is of great physical interest because it is isomorphic to domain-wall models of $p \times 1$ commensurate–incommensurate phase transitions[63,64] in two dimensions.

REFERENCES

1. A. E. Ferdinand and M. E. Fisher, *Phys. Rev.* **185**:832 (1969).
2. M. E. Fisher, in *Critical Phenomena*, M. S. Green, ed. (Academic Press, New York, 1971), p. 1.
3. Y. Imry and D. J. Bergman, *Phys. Rev. A* **3**:1416 (1971).
4. M. E. Fisher and M. N. Barber, *Phys. Rev. Lett.* **28**:1516 (1972).
5. K. Binder, *Z. Phys. B* **43**:119 (1981).
6. E. Brézin, *J. Phys.* (Paris) **43**:15 (1982).
7. M. N. Barber, in *Phase Transitions and Critical Phenomena*, Vol. VIII, C. Domb and J. L. Lebowitz, eds. (Academic Press, New York, 1983), p. 145.
8. P. Kleban and G. Akinci, *Phys. Rev. B* **28**:1466 (1983); *Phys. Rev. Lett.* **51**:1058 (1983).
9. V. Privman and M. E. Fisher, *J. Stat. Phys.* **33**:385 (1983).
10. K. Binder, M. Nauenberg, V. Privman, and A. P. Young, *Phys. Rev. B* **31**:1498 (1985).
11. K. Binder, *Z. Phys. B* **61**:13 (1985).
12. J. Zinn-Justin and E. Brézin, *Nucl. Phys. B* **257** [FS14]:867 (1985).
13. J. M. Luck, *Phys. Rev. B* **31**:3069 (1985).
14. T. W. Burkhardt and B. Derrida, *Phys. Rev. B* **32**:7273 (1985).
15. P. Kleban, G. Akinci, R. Hentschke, and R. Brownstein, *J. Phys. A* **19**:437 (1986).
16. K. Binder, *Ferroelectrics* **73**:43 (1987).
17. J. L. Cardy, in *Phase Transitions and Critical Phenomena*, Vol. XI, C. Domb and J. L. Lebowitz, eds. (Academic Press, New York, 1987), p. 55.
18. K. Binder, *Rep. Prog. Phys.* **50**:783 (1987).
19. H. Bernreuther and M. Geckeler, *Nucl. Phys. B* [FS] (1988), in press.
20. M. E. Fisher, *Rev. Mod. Phys.* **46**:597 (1974).
21. R. M. Hornreich, M. Luban, and S. Shtrikman, *Phys. Rev. Lett.* **35**:1678 (1975).
22. R. M. Hornreich and A. D. Bruce, *J. Phys. A* **11**:595 (1978).
23. G. S. Grest and J. Sak, *Phys. Rev. B* **17**:3607 (1978).
24. K. Kawasaki, in *Phase Transitions and Critical Phenomena*, Vol. II, C. Domb and M. S. Green, eds. (Academic Press, New York, 1972), p. 443.

25. S. Katz, J. L. Lebowitz, and H. Spohn, *Phys. Rev. B* **28**:1655 (1983).
26. S. Katz, J. L. Lebowitz, and H. Spohn, *J. Stat. Phys.* **34**:497 (1984).
27. J. Marro, J. L. Lebowitz, H. Spohn, and M. H. Kalos, *J. Stat. Phys.* **38**:725 (1985).
28. J. L. Vallés and J. Marro, *J. Stat. Phys.* **43**:441 (1986).
29. J. L. Vallés and J. Marro, *J. Stat. Phys.* **49**:89 (1987).
30. J. Marro and J. L. Vallés, *J. Stat. Phys.* **49**:121 (1987).
31. J. Marro, J. L. Vallés, and J. M. González-Miranda, *Phys. Rev. B* **35**:3372 (1987).
32. M. Q. Zhang, J.-S. Wang, J. L. Lebowitz, and J. L. Vallés, *J. Stat. Phys.* (1988).
33. M. Q. Zhang, Ph.D. Thesis, Rutgers University, New Brunswick, New Jersey (1987), unpublished.
34. H. K. Janssen and B. Schmittmann, *Z. Phys. B* **64**:503 (1986).
35. K.-T. Leung and J. Cardy, *J. Stat. Phys.* **44**:567 (1986).
36. W. Kinzel, in *Percolation Structures and Processes*, G. Deutscher, R. Zallen, and J. Adler, eds. (Hilger, Bristol, 1983), p. 425.
37. W. Kinzel and J. M. Yeomans, *J. Phys. A* **14**:*L*163 (1981).
38. A. M. Szpilka and V. Privman, *Phys. Rev. B* **28**:6613 (1983).
39. V. Privman, *J. Phys. A* **18**:*L*63 (1985); G. Forgacs and V. Privman, *J. Stat. Phys.* **49**:1165 (1987).
40. E. Barouch, B. M. McCoy, and T. T. Wu, *Phys. Rev. Lett.* **31**:1409 (1973), and references therein.
41. D. J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena* (McGraw-Hill, New York, 1987).
42. J. Rudnick, G. Gaspari, and V. Privman, *Phys. Rev. B* **32**:7594 (1985).
43. M. Luban, in *Phase Transitions and Critical Phenomena*, Vol. V, C. Domb and M. S. Green, eds. (Academic Press, New York, 1976), p. 35.
44. H.-K. Janssen and B. Schmittmann, private communication.
45. K. Binder, ed., *Monte Carlo Method in Statistical Physics* (Springer, Berlin, 1979).
46. M. Suzuki, *Prog. Theor. Phys.* **58**:1142 (1977).
47. P. C. Hohenberg and B. I. Halperin, *Rev. Mod. Phys.* **49**:435 (1977).
48. S. Wansleben and D. P. Landau, *J. Appl. Phys.* **61**:3968 (1987), and references therein.
49. R. H. Swendsen and J.-S. Wang, *Phys. Rev. Lett.* **58**:86 (1987).
50. L. Onsager, *Phys. Rev.* **65**:117 (1944).
51. S. Wansleben, *Computer Phys. Comm.* **43**:375 (1987).
52. T. T. Wu, *Phys. Rev.* **149**:380 (1966).
53. W. Selke, *Z. Phys. B* **43**:335 (1981).
54. S. Dietrich, in *Phase Transitions and Critical Phenomena*, Vol. XII, C. Domb and J. L. Lebowitz, eds. (Academic Press, New York, 1988), p. 1.
55. R. Lipowsky and G. Gopper, *Phys. Rev. B* **29**:5213 (1984).
56. V. Privman and N. M. Svrabic, *Phys. Rev.* **37**:3713 (1988).
57. M. Gelfand and R. Lipowsky, *Phys. Rev.* **36**:8725 (1987); H. Nabanishi and M. E. Fisher, *J. Chem. Phys.* **78**:3279 (1983).
58. D. M. Knoll and G. Gompper, KFA Jülich, preprint.
59. W. Selke, D. A. Huse, and D. M. Kroll, *J. Phys. A* **17**:3019 (1984).
60. K. Binder and D. P. Landau, *J. Appl. Phys.* **57**:3306 (1985); *Phys. Rev. B* **37**:1745 (1988).
61. S. M. Bhattacharjee and J. F. Nagle, *Phys. Rev. A* **31**:3199 (1985).
62. P. W. Kasteleyn, *J. Math. Phys.* **4**:287 (1963).
63. H. J. Schultz, *Phys. Rev. B* **22**:5274 (1980).
64. D. A. Huse and M. E. Fisher, *Phys. Rev. B* **29**:239 (1984).